IMAGINARY
Proposal for a guided tour of the exhibition

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May 7, 2012

1 Introduction and general information

1.1 General rules for the tour

1. Introduce yourself and tell people a little bit about your background. "Hi, my name ist John and I'm a final year mathematics student at University ...".

2. Mention that the exhibition is designed by the "Mathematisches Forschungsinstitut Oberwolfach", an Institut for basic mathematical research based in the Black Forest in Germany. Was first organized for the "Year of Mathematics" 2008. Also introduce the local partner or organizer (university, museum...)

3. Has been to many places up to now, e.g. several cities in Germany, but also Austria, Spain, Russia and even South America.

4. Tours should be relaxed, informal and interactive. We aim to encourage curiosity in mathematics and motivate, not bore. Thus, it is important that, when showing the exhibition, you explain what most interests you. There is not one "correct" way to guide a tour.

5. Mention the exhibition website: www.imaginary-exhibition.com:

   - All programs are available for download, encourage visitors to try at home!
   - There is also more background information, as well as competitions and other news.

6. Group tours vs. individual tours: For (school) groups it is especially important to keep the whole group interested. Take through the exhibition in a sequential manner, adapt to the level of the group. Individual tours: Easier, respond to individual interest of the visitor, take to those stations of most interest. However, you may need to actively encourage people to view the exhibition (i.e. observe the visitor and approach hin her, offer to explain something).

1.2 About the exhibition

- Imaginary aims to transmit abstract concepts of mathematics and make them comprehensible for a general audience.

- Main part of the exihibition is a display of images of algebraic surfaces (algebra – formula vs. surface – geometry)

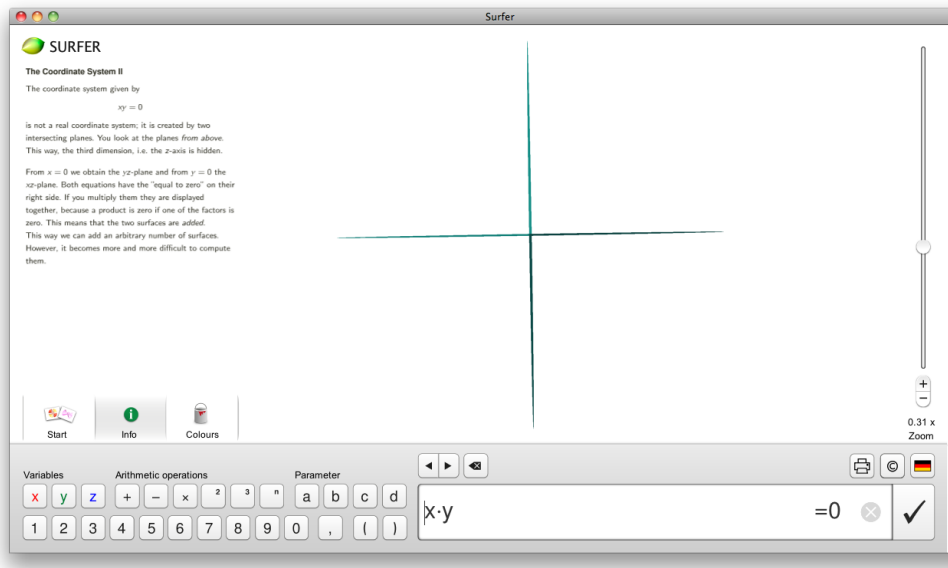- Surfer is a program to visualize and explain these surfaces

2   Surfer (depending on prerequisites)

Using this computer software and this touch screen, we are going to draw figures using our fingers. Then it is possible to print these images and visitors can take them home or stick them to the pin board.
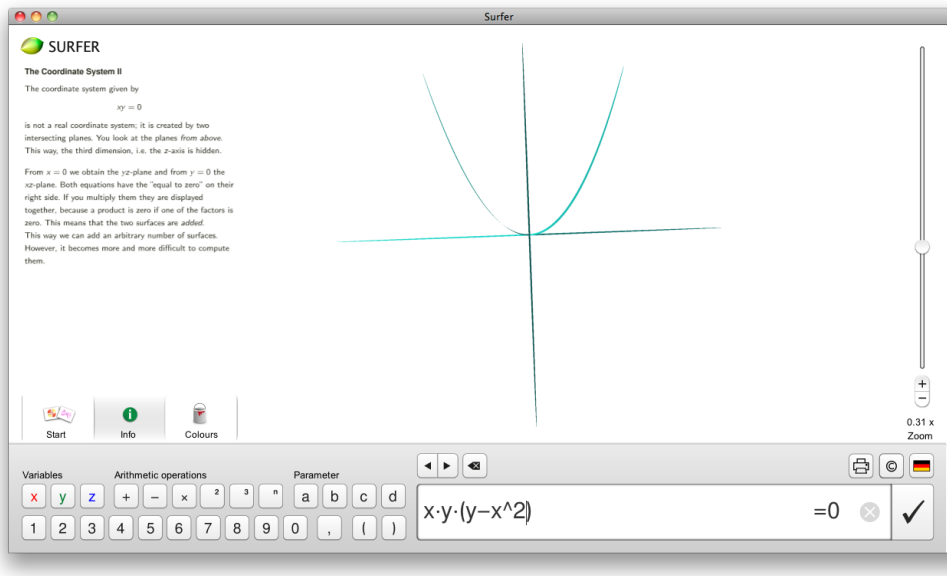
There are different ways of playing with figures. On the one hand, one can enter the formula and update the picture. On the other hand, it is possible to use shapes that have already been created and are on the left-hand side gallery.

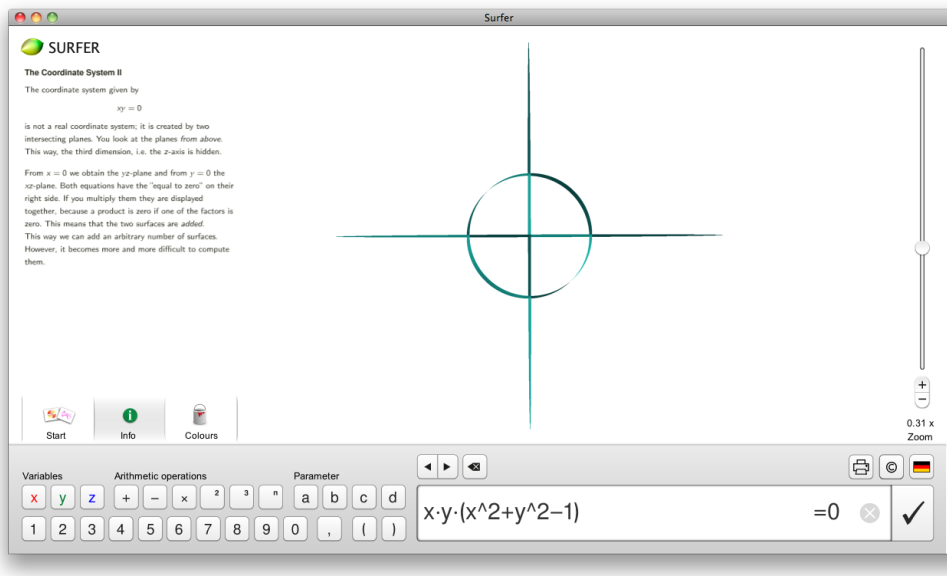The following steps can be useful, especially when showing surfer to (school) groups:

- Start with a two-dimensional coordinate system (rotate such that the z-axis is not visible; $x \cdot y = 0$)
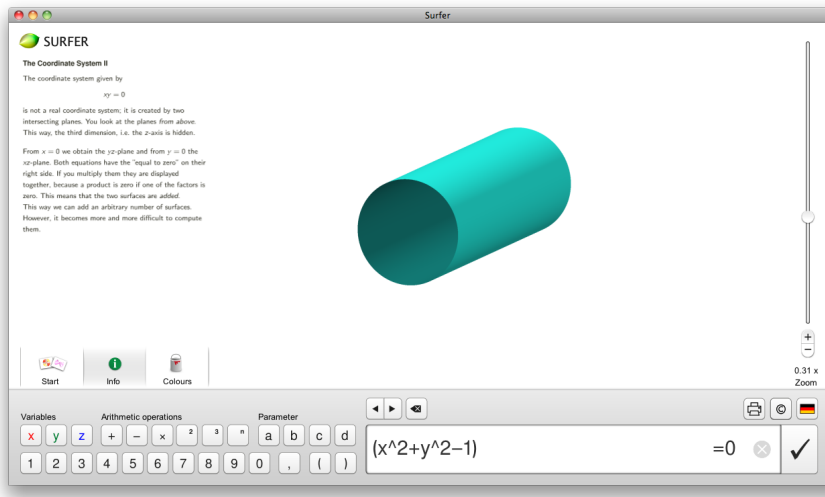
- Draw a parabola: $y = x^2 \iff y - x^2 = 0$

Surfer

SURFER

The Coordinate System II

The coordinate system given by

$$xy = 0$$

is not a real coordinate system; it is created by two intersecting planes. You look at the planes *from above*. This way, the third dimension, i.e. the z-axis is hidden.

From $x = 0$ we obtain the yz-plane and from $y = 0$ the xz-plane. Both equations have the "equal to zero" on their right side. If you multiply them they are displayed together, because a product is zero if one of the factors is zero. This means that the two surfaces are *added*. This way we can add an arbitrary number of surfaces. However, it becomes more and more difficult to compute them.

Start Info Colours

0.31 x Zoom

Variables: x y z Arithmetic operations: + − × ² ³ ⁿ Parameter: a b c d
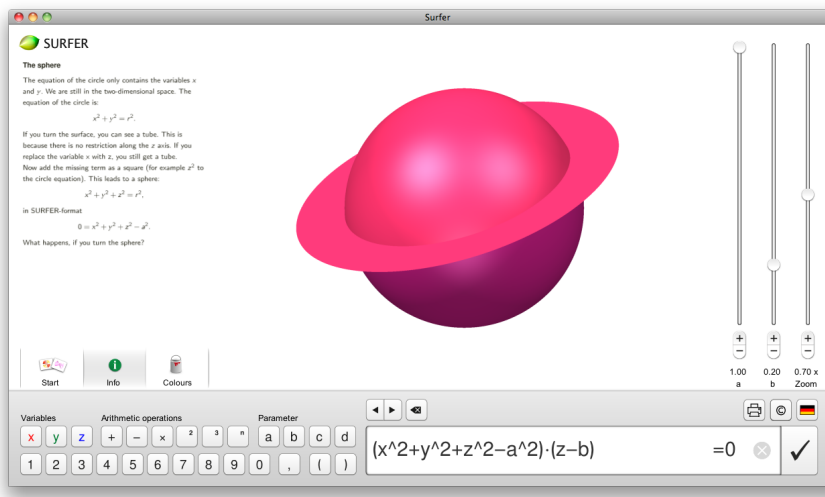1 2 3 4 5 6 7 8 9 0 , ( )

$x·y·(y−x^2)$ =0

• Draw a circle: $x^2 + y^2 = 1 \iff x^2 + y^2 - 1 = 0$ (If necessary, start by explaining the equation of a circle with the help of Pythagoras' theorem), then delete the coordinate system, rotate the surface and show that the values in z-direction are not bounded, i.e. you get a cylinder.

Surfer

SURFER

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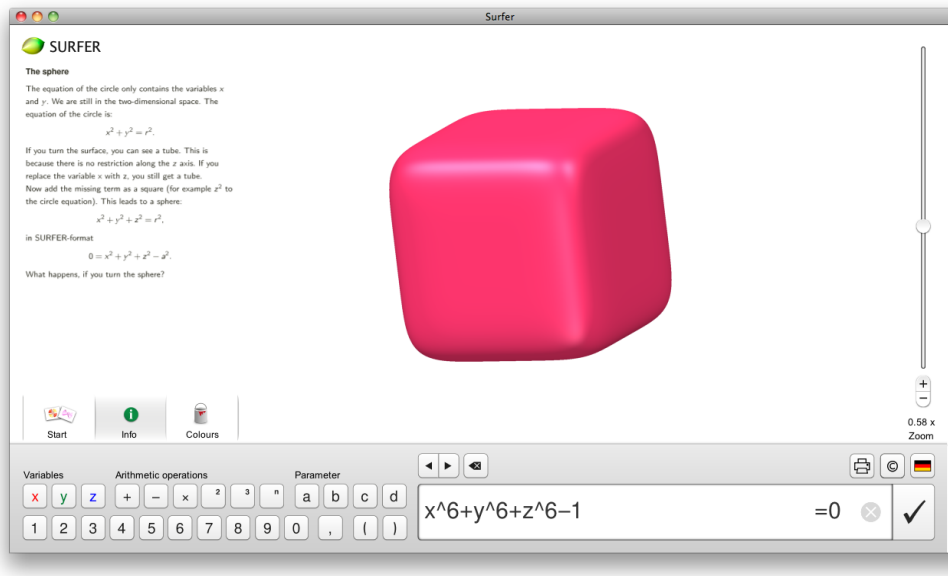1 2 3 4 5 6 7 8 9 0 , ( )

$x·y·(x^2+y^2−1)$ =0

- Let's switch to 3D: In analogy to the circle, we are now looking at a sphere: $x^2 + y^2 + z^2 - 1 = 0$. Explain how the sphere is generated: Choose explicit values for the z-coordinate, e.g. $z = 0, 1, -1$ and show the result by intersecting the sphere with a suitable plane (equator, north pole, south pole). It is now very amusing to make use of the parameters a and b. If you replace the -1 in the formula by $-a^2$ and vary the parameter, you can observe that the radius of the sphere changes. Multiplying the equation of the sphere by $(z - b)$ and then playing along the b parameter, a disk goes up and down the ball and one can see very nicely that the sphere is built out of circles of different radii. For $b = 0$ we get a saturn-shape, or mexican sombrero if seen from above.

What happens if we change the powers? With z^3 we have a bell-shape and going on like this it is easy to find a connection between a sphere and a cube: try for instance a high even exponent such as six.

This can be a good moment to motivate playing around with surfer. As you can imagine, it is very difficult to draw shapes one has in mind, but with this software one can learn a lot of mathematics. The following techniques can be helpful to build nice surfaces:

- Melting and de–melting:

  Adding b to the equation of the Saturn $y \cdot \left(x^2 + y^2 + z^2 - 1\right)$ and playing with the value of the parameter, it is possible to melt the top and separate the bottom of the planet and vice versa.

- Fading:

  It is also possible to "fade" between two shapes. If f and g denote the equations of the two surfaces, you can fade between them, by looking at $af + (1 - a)g$ and changing the value of the parameter. For example, let's convert a sphere in to a cube:

  $$a(x^2 + y^2 + z^2 - 1) + (1 - a)(x^6 + y^6 + z^6 - 1).$$

  For $a = 1$, we get the sphere, for $a = 0$ the cube and for any other value of a something in between.

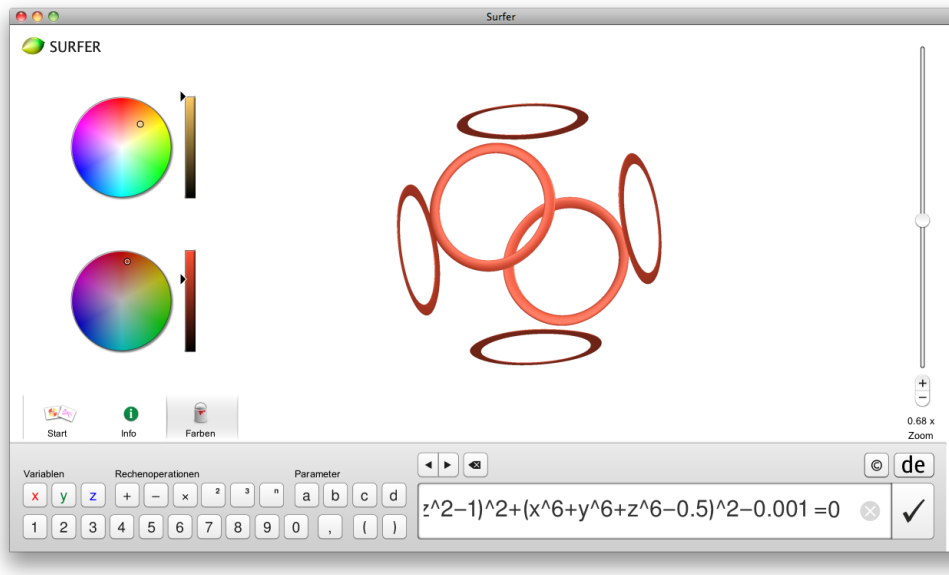- Union and Intersection of two shapes:

  If we want to display two surfaces at the same time, we just need to multiply their formulas. A product is zero if at least one of the factors is zero, so the roots of the product are the union of the roots of the factors.

  To display the intersection of the two shapes we need to find all points were both formulas are zero simultaneously. This is done by adding the squares of the formulas. However, to make the infinitely thin intersection curve visible, we need to cheat a little bit and subtract

a small constant from the equation. Here is an example for intersecting a sphere of radius 1 and a cube of radius 0.5:

$$(x^2 + y^2 + z^2 - 1)^2 + (x^6 + y^6 + z^6 - 0.5)^2 - 0.001$$

- Singularities:

  Visually speaking, singularities are parts of the surface which look very pointed. A good example is the double cone: like the sphere, the double cone is built out of circles with varying radii (however, in contrast to the sphere, the radius is varying linearly and not quadratically). In the middle, at the intersection of the two cones, the surface has a singularity. Analytically speaking, the derivative (the gradient) of the corresponding function vanishes at this point.

  When introducing Surfer to individual people or people with a background in mathematics, it can be very interesting to mention the theory and problems related to singularities. In the case of a cone, which has one singularity, changing the a parameter can have catastrophic consequences. As we change a from 0.5 to 0.49, the two cones separate from each other. If instead we increase the value to 0.51, the two cones are merged and the singularity vanishes.

- World record surfaces:

  The degree of a surface of the order of the corresponding polynomial. For every degree d, there is a the maximal amount of singularities $\mu(d)$. For polynomials of degree 6 the record is 65: the world record is the Barth Sextik, and the inventor also proved that this world record is the maximum number one can attain. In the case of polynomials of degree 7, the record is 99 but the theoretical upper bound is 104. However, $\mu(7)$ is not yet known.

- Interaction with the visitors

  The main target of the tour is not showing, but enabling and encouraging the visitors to play around with the programs themselves. An easy approach to this is starting with an existing

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surface from the gallery and changing its appearance by modifying its formula, e.g. putting the heart into a pair of trousers by changing the last z-exponent from 3 to 2. Alternatively, one can also start from the scratch, adding terms at random and watching the surface change. This is usually a good ending point of the tour.