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Friezes and tilings

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Friezes have occured as architectural ornaments for many centuries. In this snapshot, we consider the mathematical analogue of friezes as introduced in the 1970s by Conway and Coxeter. Recently, infinite versions of such friezes have appeared in current research. We are going to describe them and explain how they can be classified using some nice geometric pictures.

1 Friezes

Friezes have been used since antiquity as stylistic ornaments in architecture and decorative art. Figure 1 shows some examples of drawings of such friezes.

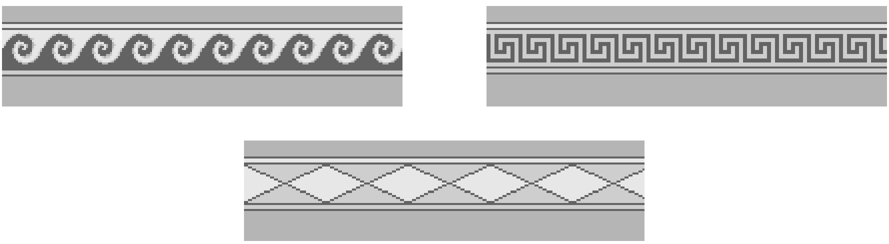


Figure 1: Friezes as decorative elements.

In this snapshot, we want to investigate mathematical versions of these ornaments, called *frieze patterns*, which are built up from numbers. These go back to the famous mathematicians John Horton Conway (born 1937) and

Figure 2: John Horton Conway *(left)* (born 1937) and Harold Scott MacDonald Coxeter *(right)* (1907 – 2003).

Harold Scott MacDonald Coxeter (1907 – 2003) (see Figure 2), who introduced and studied them in the early 1970s [5]. Before we are going to define frieze patterns properly, you might want to have a look at some first examples in Figure 3 (the green [*italic*] and red [**bold**] colours will be explained later).

1 1 1 1 1 1 1 1 1 1 1 1 1 1

$\cdots$ 2 1 2 1 2 1 2 1 2 1 2 1 2 1 2 $\cdots$

1 1 1 1 1 1 1 1 1 1 1 1 1 1

(a) A simple frieze pattern.

1 1 1 *1* **1** 1 1 1 1 1 1 1 1 1 1 1 1

$\cdots$ 2 2 1 4 *2* **1** 4 1 3 2 1 4 2 1 4 1 3 2 $\cdots$

5 1 3 7 *1* **3** 3 2 5 1 3 7 1 3 3 2 5

$\cdots$ 3 2 2 5 3 *2* **2** 5 3 2 2 5 3 2 2 5 3 2 $\cdots$

1 3 3 2 5 *1* **3** 7 1 3 3 2 5 1 3 7 1

$\cdots$ 2 1 4 1 3 2 *1* **4** 2 1 4 1 3 2 1 4 2 1 $\cdots$

1 1 1 1 1 1 *1* **1** 1 1 1 1 1 1 1 1 1

(b) A more complicated frieze pattern.

Figure 3: First examples of frieze patterns.

Similar to the architectural ornaments above, frieze patterns are infinite horizontal arrays; they consist of a finite number of rows of positive integers and they are bounded by two rows of 1's. The crucial condition for a frieze pattern is that every diamond-shaped set of four adjacent numbers within the pattern

$$\begin{matrix} & b & \\ a & & d \\ & c & \end{matrix}$$

satisfies the relation $ad - bc = 1$. We call it the *determinant condition* for now.[1]

With the concept and some first examples at hand (you are invited to find some more frieze patterns yourself!), a mathematician would be tempted to try to get an overview over all such objects, if possible. In fact, a main result from Conway and Coxeter's research states that every frieze pattern can be obtained geometrically via *triangulations* of polygons. [2]

A triangulation of a regular n-gon is a collection of *diagonals* (straight lines between two non-neighbouring vertices) which do not cross, such that the interior of the n-gon is divided into triangles. In Figure 4, you find an example of a triangulation of a regular octagon. It is not hard to see that each triangulation of an n-gon contains $n-3$ diagonals and divides the n-gon into $n-2$ triangles.

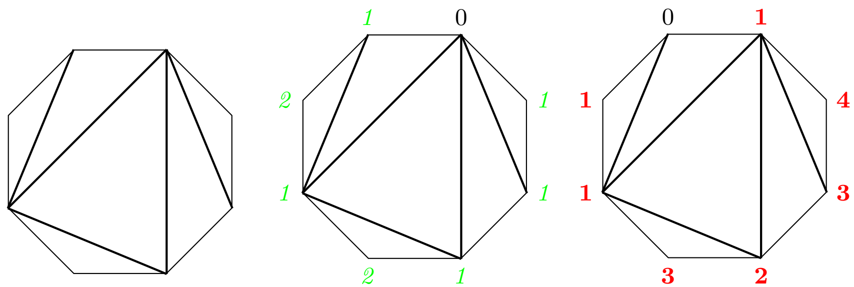


Figure 4: A triangulation of the octagon.

[1] If you are familiar with determinants you might have noticed that this just says that the determinant of the 2×2-matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ equals 1.

[2] A *polygon* is a geometrical figure in the plane bounded by a closed chain of line segments (*sides* or *edges* of the polygon) such as a triangle, a square, etc. The points, where the edges meet are called *corners* or *vertices*. An *n-gon* is a polygon with exactly n vertices. A polygon is said to be *regular* if all edges have the same length.

How could one produce a frieze pattern from such a triangulation? There is the following simple recipe (see [4]), which uses nothing more difficult than primary school mathematics.

The primary school algorithm: Pick a vertex of the polygon, and assign the value 0 to it. Next, assign the value 1 to each vertex sharing a triangle with the given vertex. Then, inductively, whenever two vertices of a triangle have already been assigned numbers a and b, say, the third vertex gets assigned their sum $a + b$.

Consider the triangulation of the regular octagon in Figure 4. In the middle and on the right we have computed the numbers as above, starting with the top two vertices.

Now, reading the non-zero numbers counter-clockwise gives a diagonal in the corresponding Conway-Coxeter frieze pattern; and putting the diagonals for the n vertices next to each other (in subsequent order, and repeated after every n steps), indeed gives a frieze pattern. In our example in Figure 4, we obtain the green [*italic*] and red [**bold**] slice in the above Conway-Coxeter frieze pattern shown in Figure 3b.

Indeed, as Conway and Coxeter found out, every frieze pattern stems from a triangulation of a regular polygon in this way, and vice versa. This fact hence provides a nice classification of all frieze patterns through geometric objects and simultaneously links frieze patterns to various other mathematical objects which are related to triangulations; for more details see [6, p. 96-100].

Also, if we are able to count the number of triangulations of an n-gon, Conway and Coxeter's result tells us immediately how many frieze patterns with $n - 1$ rows exist. So let's count polygon triangulations:

- for a triangle there is only one triangulation (namely the 'empty one' with no diagonal);
- we can find two triangulations for a quadrilateral;
- a pentagon has five possible triangulations;
- for a hexagon there already exist 14 different triangulations! (Can you find them all?)

Arranging these numbers in a series, we obtain: $1, 2, 5, 14 \ldots$. Enter these four numbers into an internet search engine – you'll be immediately pointed to a well-known sequence of integers, the so-called *Catalan numbers*.

Of course, the observation that triangulations of n-gons are counted by the Catalan numbers for the four smallest $n = 3, 4, 5, 6$ suggests that one could extend our list above even for larger numbers n. In fact, this is true for arbitrary n. For a proof of this conjecture and if you are interested in other mathematical contexts where Catalan numbers appear, see Richard Stanley's book [9], or the additional material provided on his web page [8].

2 Tilings

We now come to very recent research topics which were inspired by the Conway-Coxeter frieze patterns. Actually, it needs nothing more than a simple removal of the condition of having limiting rows of 1's in frieze patterns to obtain new and interesting mathematical objects. These are called *tilings*.

A tiling is a pattern of infinitely many rows of infinitely many positive integers, satisfying the determinant condition for every adjacent diamond of numbers [3] just like above. The notion of tilings thus naturally extends that of Conway-Coxeter frieze patterns from finite to infinite (when considering the number of rows). In the literature on tilings, it is customary to arrange the entries of rows into columns. In other words, contrary to the frieze patterns shown in Figure 3, we write the numbers on top of each other, in order to be consistent with the literature. In Figure 5, you find an example of such a tiling.

Note that the entries in this example are given by every second one of the famous Fibonacci numbers. The Fibonacci numbers are defined by $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_{n-1} + F_n$ for all $n \geq 1$; the determinant condition for

$$\begin{array}{ccccccccccc}
 & & & & & \vdots & & & & & \\
 & 1597 & 610 & 233 & 89 & 34 & 13 & 5 & 2 & 1 & \\
 & 610 & 233 & 89 & 34 & 13 & 5 & 2 & 1 & 1 & \\
 & 233 & 89 & 34 & 13 & 5 & 2 & 1 & 1 & 2 & \\
 & 89 & 34 & 13 & 5 & 2 & 1 & 1 & 2 & 5 & \\
\cdots & 34 & 13 & 5 & 2 & 1 & 1 & 2 & 5 & 13 & \cdots \\
 & 13 & 5 & 2 & 1 & 1 & 2 & 5 & 13 & 34 & \\
 & 5 & 2 & 1 & 1 & 2 & 5 & 13 & 34 & 89 & \\
 & 2 & 1 & 1 & 2 & 5 & 13 & 34 & 89 & 233 & \\
 & 1 & 1 & 2 & 5 & 13 & 34 & 89 & 233 & 610 & \\
 & & & & & \vdots & & & & &
\end{array}$$

Figure 5: A simple tiling, involving Fibonacci numbers.

[3] Of course, after rearranging the rows into columns as in Figure 5, the diamonds become ordinary squares.

the tiling in Figure 5 then reads $F_{m+2}F_{m-2} - F_m^2 = 1$ for odd m. That this identity is indeed true follows directly from the Cassini (or Catalan) identity, one of the many properties Fibonacci numbers satisfy.

Tilings as described in the above paragraphs were recently introduced by Ibrahim Assem, Christophe Reutenauer, and David Smith [1] by the name of *SL_2-tilings*. They appeared in their work on the fascinating new theory of cluster algebras and turned out to be useful for developing certain formulas in this research area. For some background on cluster algebras we refer to the article [10] by Andrei Zelevinsky.

While the Fibonacci number tiling of Figure 5 was a rather simple example, more complicated examples of tilings can be constructed, see for instance Figure 6 (again the meaning of the colouring [**bold print**] will be explained later). You are invited to produce more tilings yourself!

					⋮					
	265	218	171	124	77	107	137	167	197	
	203	167	131	95	59	82	105	128	151	
	141	116	91	66	41	57	73	89	105	
	79	65	51	37	23	32	41	50	59	
⋯	**17**	**14**	**11**	**8**	**5**	**7**	**9**	**11**	**13**	⋯
	57	47	37	27	17	24	31	38	45	
	154	127	100	73	46	65	84	103	122	
	405	334	263	192	121	171	221	271	321	
					⋮					

Figure 6: A more complicated tiling.

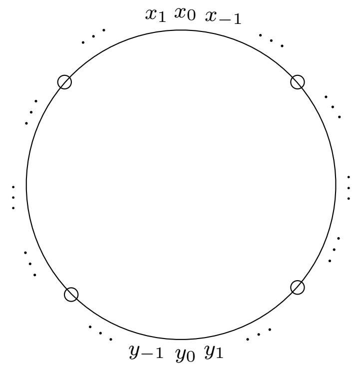
A mathematician would now immediately ask whether there is a nice and structured way to find and describe all of them, that is, he or she would try to *classify* all tilings. This is actually the same question we asked for the frieze patterns; there we learned that triangulating polygons produces all frieze patterns.

And indeed, YES, also tilings can be classified, as shown recently in joint work with Christine Bessenrodt and Peter Jørgensen [3]. It turns out that, as for Conway-Coxeter frieze patterns, one can produce all tilings by triangulating certain geometrical objects – not polygons but more complicated ones – still using the same primary school algorithm.

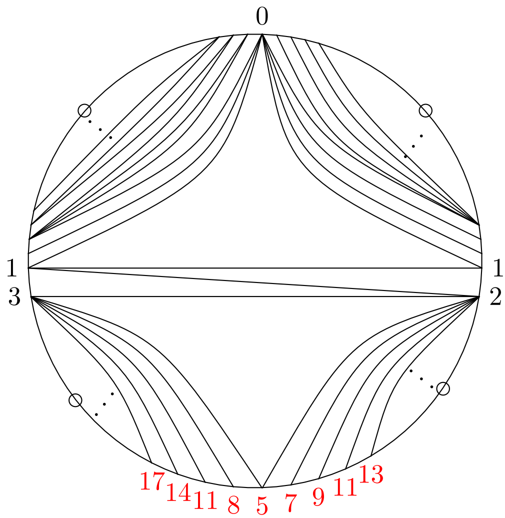
We now describe those new mathematical objects, see Figure 7a: We take a circle, and mark four distinguished points on it; they divide the circle into four regions. In each of these regions we insert a set of infinitely many vertices, indexed by the integers. For the top and bottom region these vertices are named $\ldots, x_{-1}, x_0, x_1, \ldots$ and $\ldots, y_{-1}, y_0, y_1, \ldots$, respectively, in Figure 7a.

We now have an object with vertices which allows us to perform a triangulation on it by drawing lines between the vertices – just as for the polygons before. This is illustrated in Figure 7b for the example tiling shown in Figure 6. [4]

On any such triangulation, we perform the primary school algorithm between the vertices in the top and the bottom regions of the circle. That is, we start from any vertex in the top region and assign a 0 to it. We then read off all the numbers from the vertices in the bottom region produced by the algorithm. In the triangulation of Figure 7b, the red [**bold**] numbers at the bottom are obtained from the primary school algorithm and they indeed give the red [**bold**] row of the tiling in Figure 6. The other rows of the tiling in Figure 6 are then obtained by performing the same algorithm, starting with every other vertex in the top region of the circle.



(a) A circle with four disinguished points.



(b) The primary school algorithm performed on a triangulation.

Figure 7

[4] Note that in Figure 7b, the triangles now of course do no longer look like ordinary triangles with straight edges.

It mainly follows from classical Conway-Coxeter theory that starting from the vertices at the top, the numbers you obtain in the bottom region indeed give the rows of a tiling. The hard part of our result is to show that, conversely, for every tiling one can indeed find a suitable triangulation of the circle with four distinguished points which produces the given tiling using the familiar primary school algorithm.

The proof of this fact is constructive: the diagonals of the triangulation to be found can be read off from the entries of the given tiling (in a rather subtle and non-obvious way, though). The methods used in the proof are completely different for the cases where the tiling contains 1's and the much harder case where it does not. For the latter case, a crucial observation for getting started is that any tiling without 1's has the property that its minimal entry appears exactly once; see for instance the tiling in Figure 6 where indeed there is the unique minimal entry 5.

3 Concluding remarks

As is often the case in mathematics, the work on solving one problem creates new questions and inspires future research also here. In the theory of frieze patterns and tilings there are several variations and generalisations which are currently being studied.

One obvious generalisation is to relax the condition allowing only positive integers as entries. Interesting frieze patterns and tilings with arbitrary integer entries exist and research in this direction has just begun, see for instance [7].

Another area of current research are the so-called SL_k-tilings, introduced in [2]. These are arrays of positive integers such that each adjacent $k \times k$-determinant equals 1. The tilings considered in this snapshot arise as the special case $k = 2$. For SL_k-tilings, a nice geometric description like the one given above for SL_2-tilings is not yet known.

From all these contexts, one can expect that new and interesting mathematical objects and combinatorial methods might wait to be discovered, and hopefully also some surprising new connections to other areas of mathematics.

Image credits

Figure 1 top left image; “Laufender Hund”. Author: Roland Bergmann. Licensed under Creative Commons Attribution-Share Alike 3.0 via Wikimedia Commons, http://commons.wikimedia.org/wiki/File%3ALaufender_Hund.png [Online; accessed 09-December-2014]

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References

[1] I. Assem, C. Reutenauer, and D. Smith, *Friezes*, Adv. Math. **225** (2010), 3134–3165.

[2] F. Bergeron and C. Reutenauer, SL_k*-tilings of the plane*, Illinois J. Math. **54** (2010), 263–300.

[3] C. Bessenrodt, T. Holm, and P. Jørgensen, *All* SL_2 *tilings come from triangulations,* research report for the Oberwolfach meeting on ’Cluster algebras and related topics’, December 2013, available at http://www.iazd.uni-hannover.de/~tholm/ARTIKEL/SL_2_Report_OW_Dec2013.pdf, [Online; accessed 27-November-2014].

[4] D. Broline, D. W. Crowe, and I. M. Isaacs, *The geometry of frieze patterns*, Geom. Dedicata **3** (1974), 171–176.

[5] J. H. Conway and H. S. M. Coxeter, *Triangulated polygons and frieze patterns*, Math. Gaz. **57** (1973), 87–94 and 175–183.

[6] J. H. Conway and R. K. Guy, *The book of numbers*, Copernicus, New York, 1996.

[7] S. Morier-Genoud, V. Ovsienko, and S. Tabachnikov, $SL_2(\mathbb{Z})$*-tilings of the torus, Coxeter-Conway friezes and Farey triangulations*, arxiv:1402.5536v1, 2014.

[8] R. Stanley, *Catalan addendum*, http://www-math.mit.edu/~rstan/ec/catadd.pdf, [Online; accessed 26-November-2014].

[9] ———, *Enumerative combinatorics*, Cambridge Studies in Advanced Mathematics, vol. 2, Cambridge University Press 62, Cambridge, 1999.

[10] A. Zelevinsky, *What is ... a cluster algebra?*, Notices Amer. Math. Soc. **54** (2007), 1494–1495.

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