Snapshots of modern mathematics
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Footballs and donuts in four dimensions

Steven Klee

In this snapshot, we explore connections between the mathematical areas of counting and geometry by studying objects called simplicial complexes. We begin by exploring many familiar objects in our three dimensional world and then discuss the ways one may generalize these ideas into higher dimensions.

1 Counting and geometry in three dimensions

Imagine a football. We begin by restricting our attention to the iconic Adidas Telstar ball, which is constructed by stitching together a collection of leather patches (see Figure 1). Some of these patches are shaped like hexagons and others are shaped like pentagons.

Objects such as this football are studied in the area of geometric combinatorics. Combinatorics is the mathematical science of counting, and geometric combinatorics applies these counting techniques to geometric objects.

What can we count on a football? First of all, we use patches to construct the ball. Let's call the number of patches F (which stands for *faces*). In addition, as we stitch our ball together, we create some seams where two patches come together. Let's call that number of seams E (which stands for *edges*). Finally, there are some points where many patches mutually meet. Let's call that number of meeting points V (which stands for *vertices*).

The Telstar football has 12 pentagonal faces and 20 hexagonal faces, so $F = 32$. We can also count that $E = 90$ and $V = 60$. How could we do this? To count the edges, first we notice that each edge is formed as the result of two

Figure 1: The Adidas Telstar ball.

patches being stitched together. Therefore, each pentagonal patch contributes to five stitches and each hexagonal patch contributes to six stitches. This means the patches contribute to $12 \times 5 + 20 \times 6 = 180$ stitches. But each edge has been counted twice here – once for each of the contributing patches – so the total number of edges is $180 \div 2 = 90$. As for counting the vertices, we can notice that each vertex is the meeting point of one black pentagonal patch and two white hexagonal patches. Each pentagonal patch contributes 5 vertices to the ball. Since there are 12 pentagonal patches, there must be a total of $5 \times 12 = 60$ vertices.

An important quantity to consider in the area of geometric combinatorics is the *Euler characteristic*[1], which is defined as the sum $\chi = V - E + F$. For the Telstar football, $\chi = 60 - 90 + 32 = 2$.

Now suppose we change the rules for how we construct a football. Instead of using only hexagonal and pentagonal patches, we might want to use patches shaped like other polygons, such as triangles, squares, octagons, or decagons. The only constraint is that we still must stitch all these patches together to form a football.[2] As an example, we could have started with 6 square patches that would be stitched together to form a cube. In this case, $F = 6$, $E = 12$, and $V = 8$. Once again, $V - E + F = 2$.

In general, it still makes sense to count the number of vertices, edges, and faces in any football we could possibly stitch together out of polygonal patches.

[1] Named after the famous Swiss mathematician Leonhard Euler (1707–1783) who contributed immensely to the study of polyhedra.

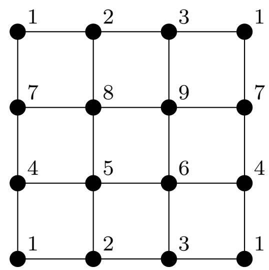
[2] FIFA may not be excited to use the resulting ball in the next World Cup. But we are still interested in these objects mathematically.

The most fundamental result in this field states that no matter how we choose to stitch these patches together[3], no matter how many patches we use, and no matter how nonsensical it would be to use the resulting ball, it will *always* be the case that $V - E + F = 2$. This fact justifies the use of the term *characteristic*. Indeed, every geometric object which is sufficiently similar to a football – that can be deformed and smoothed into the shape of a sphere without tearing or stitching new patches – will have $\chi = 2$. Proofs of this result can be found in the website *Twenty proofs of Euler's formula* [2].

1.1 From footballs to donuts

Given a collection of polygonal patches, what other geometric objects could you construct by stitching the patches together?

As an example, let us consider the following object, which is constructed from a collection of square patches.



In this case, the labels on the vertices suggest there is additional stitching that must occur. For example, the edge $(2, 3)$ on the bottom of the figure and the edge $(2, 3)$ on the top of the figure are meant to be identified. In fact, every edge around the perimeter of this figure is meant to be sewn to its corresponding edge on the opposite side of the figure.

We illustrate this process in Figure 2. We begin with a flat square on the left side of the figure. If we first sew together the edges along the top and bottom of this figure, the resulting object becomes a cylinder (Figure 2 center). If we then glue together the edges along the sides of the figure, we stitch the left side of the cylinder to its right side and obtain a *torus*, or a donut (Figure 2 right).

[3] Within reason. When stitching two patches together, we will choose an edge of the first patch and an edge of the second patch, and the patches will be sewn together along the entire length of their respective edges. In contrast, we do not allow three patches to meet along a single stitching. Nor do we allow a "half-stitching" in which an edge from one patch is stitched only to half of an edge of another patch. As such, we should think of our patches as being relatively flexible (perhaps they are made of rubber rather than leather) so that the lengths of the edges of the patches do not prevent them from being stitched together.

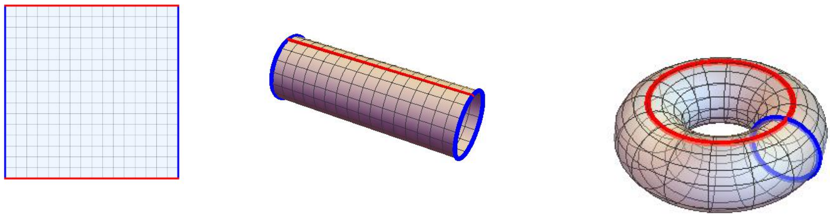


Figure 2: Forming a torus by identifying the sides of a square

Now let us compute the Euler characteristic of this torus. After we have made all of our identifications, we have 9 vertices, 18 edges, and 9 square faces, which means $V = 9$, $E = 18$, and $F = 9$. So $\chi = V - E + F = 0$. This is important because it tells us that the Euler characteristic of a torus is different than the Euler characteristic of a sphere. As a corollary, we have discovered an incredibly important fact: a torus and a sphere are fundamentally different objects. Informally, this encodes the observation that a torus has a hole in the middle, while a sphere does not. More formally, these objects are different in the sense that it is not possible to transform one into the other in a continuous manner.

2 Moving into higher dimensions

In our previous examples, the football and torus we constructed should be viewed as 2-dimensional objects, even though they live in our 3-dimensional world. The reason for this is that the polygonal patches we use are 2-dimensional.

From here, we can describe many faces of modern mathematical research. For the remainder of this note, we will describe just one area of research. For these two-dimensional objects, we might begin by restricting our attention to those objects that can be constructed by using only triangular patches.

Mathematically, we call a triangle a 2-*simplex.* One way to describe a triangle is to say that it is the *convex hull* of three non-collinear points (that is, points that do not all lie on the same line) in the plane. Informally, the convex hull of a set of points is the smallest portion of space that is enclosed by those points – imagine that the points are all connected by lines and we then color in the space inside the lines. The following diagram illustrates three possibilities for what the convex hull of four non-collinear points in the plane could look like.

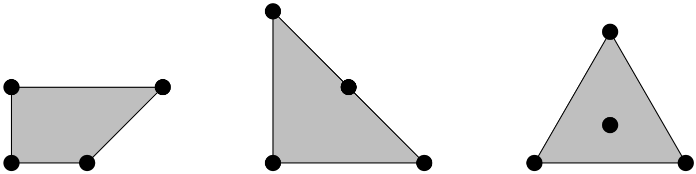


Figure 3: Examples of convex hulls

In general, a d-dimensional simplex[4] can be defined as the convex hull of a set of $(d+1)$ points in d-dimensional space that do not all lie in a common plane (or in higher dimensions, a common hyperplane). For example, a 1-dimensional simplex is a line segment (the convex hull of two points) and a 3-dimensional simplex is a tetrahedron. Similarly, a 0-dimensional simplex consists of a single point. For this reason, we refer to 0-dimensional simplices as *vertices* and 1-dimensional simplices as *edges*.

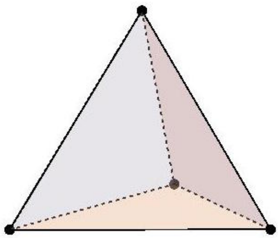


Figure 4: A tetrahedron

One important structural feature of a d-dimensional simplex is that its boundary is built from simplices of lower dimensions. For example, the boundary of a tetrahedron consists of four triangles, six edges, and four vertices. These are called the *boundary faces* of the tetrahedron. In general, a d-dimensional simplex has $d+1$ vertices and hence $\binom{d+1}{i+1}$[5] boundary faces of dimension i for each $0 \leq i \leq d-1$ because we can choose any $i+1$ of its vertices to form an i-dimensional boundary face.

[4] The plural of 'simplex' is 'simplices'.

[5] The notation $\binom{n}{k}$ tells us the number of ways to choose k objects from a set of n objects when order does not matter. This is sometimes written as ${}_nC_k$. It can also be expressed as $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

A *simplicial complex* is a geometric object that is constructed from a given set of simplices (these are analogous to the patches we used to construct our footballs earlier) with the property that any two simplices intersect either along a single one of their boundary faces (just as the patches in our footballs were stitched together along a common edge or met at a single vertex) or not at all.

If Δ is a simplicial complex, the most natural objects to count are the *face numbers* of Δ. We write $f_i(\Delta)$ to denote the number of i-dimensional simplices in Δ. For example, $f_0(\Delta)$ denotes the number of vertices in Δ; $f_1(\Delta)$ denotes the number of edges in Δ; $f_2(\Delta)$ denotes the number of triangular faces in Δ; $f_3(\Delta)$ denotes the number of tetrahedra in Δ; and so on. We typically add an extra condition stating that $f_{-1}(\Delta) = 1$, which corresponds to the $\binom{d+1}{0} = 1$ way to choose the empty set as a subset of zero vertices contained in the boundary of any simplex.

If Δ is a d-dimensional simplicial complex, we arrange its face numbers in a list called the *f-vector* of Δ, which we write as $f(\Delta) = (f_0, f_1, f_2, \dots, f_d)$. Furthermore, we define the *Euler characteristic* of Δ as

$$\chi(\Delta) = f_0 - f_1 + f_2 - \cdots + (-1)^d f_d.$$

As in the 2-dimensional case, two simplicial complexes that can be smoothly deformed to one another without tearing will have the same Euler characteristic.

In the previous section, we saw that the Euler characteristic of a two-dimensional sphere (a football) is equal to 2. In 1964 Victor Klee [3] proved a much more general result, known as the Dehn–Sommerville Equations, that extends this result to spheres of all dimensions. For concreteness, a 1-dimensional sphere is a circle and a 2-dimensional sphere is a football. In higher dimensions, we can still define a sphere as all points that lie at some fixed distance from the origin.

Theorem 1 *Let Δ be a simplicial complex that forms a d-dimensional sphere. Then for any $-1 \leq j \leq d$,*

$$(-1)^d f_j(\Delta) = \sum_{i=j}^{d} (-1)^i \binom{i+1}{j+1} f_i(\Delta). \tag{1}$$

For example, if we take $d = 2$ and $j = -1$ in Equation (1), we get

$$\begin{aligned} 1 &= \sum_{i=-1}^{2} (-1)^i \binom{i+1}{0} f_i(\Delta) \\ &= -f_{-1}(\Delta) + f_0(\Delta) - f_1(\Delta) + f_2(\Delta) \\ &= -1 + V - E + F, \end{aligned}$$

which is equivalent to saying that $\chi(\Delta) = 2$.

Similarly, taking $d = 3$ and $j = -1$ in Equation (1) tells us that the Euler characteristic of a three-dimensional sphere is equal to 0. More generally, the Euler characteristic of a d-dimensional sphere is equal to 2 when d is even and 0 when d is odd.

The Dehn–Sommerville equations mark a first major constraint that can be put on the f-vectors of simplicial spheres. Many other geometric objects, that generalize the sphere and the torus (called *manifolds*), can be smoothly deformed into simplicial complexes. Thus, they can be investigated using results such as the Dehn–Sommerville equations. For example, we can decide that two manifolds cannot be smoothly deformed from on to the other (geometric property) since they do not have the same Euler characteristic (combinatorial property). The questions that have motivated and continue to motivate modern research seek additional relationships between the combinatorics of the f-vector and the geometric structure of manifolds.

Somewhat surprisingly, even though these questions of understanding f-vectors of spheres and manifolds seem to be deeply rooted in geometry and combinatorics, many of the landmark results in this field have made significant use of machinery of other branches of mathematics. Specifically, tools from the fields of commutative algebra and algebraic geometry are essential in the modern field of geometric combinatorics.

Further reading

Messer and Straffin [4] give a very approachable introduction to topolgy, which is the mathematical discipline interested in answering questions such as why the sphere and the torus are mathematically different objects. The recent book of De Loera, Ramau, and Santos [1] provides an excellent introduction to simplicial complexes and polytopes from a theoretical and applied viewpoint. Ziegler's book [5] on polytopes is a fantastic resource for more advanced students.

Image credits

Fig. 1 "Telstar Durlast". Author: Warren Rohner. Licensed under Creative Commons Attribution-Share Alike 2.0 Generic. Retrieved from https://commons.wikimedia.org/wiki/File:Adidas_Telstar.jpg, accessed May 1, 2015.

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[1] J. A. De Loera, J. Rambau, and F. Santos, *Triangulations*, Algorithms and Computation in Mathematics, vol. 25, Springer-Verlag, Berlin, 2010, Structures for algorithms and applications.

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[4] R. Messer and P. Straffin, *Topology now!*, Classroom Resource Materials Series, Mathematical Association of America, Washington, DC, 2006.

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Steven Klee *is an assistant professor of mathematics at Seattle University.*

Mathematical subjects
Discrete Mathematics and Foundations, Geometry and Topology

Connections to other fields
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Daniel Kronberg
junior-editors@mfo.de

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Mathematisches Forschungsinstitut
Oberwolfach gGmbH
Schwarzwaldstr. 9 –11
77709 Oberwolfach
Germany

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