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# Snake graphs, perfect matchings and continued fractions

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A continued fraction is a way of representing a real number by a sequence of integers. We present a new way to think about these continued fractions using snake graphs, which are sequences of squares in the plane. You start with one square, add another to the right or to the top, then another to the right or the top of the previous one, and so on. Each continued fraction corresponds to a snake graph and vice versa, via "perfect matchings" of the snake graph. We explain what this means and why a mathematician would call this a combinatorial realization of continued fractions.

## 1 Continued fractions

Let us start with an easy example. If we are given a rational number $\frac{30}{13}$, we may ask ourselves "how big is this number?". And our first answer to that question may be, "well, it is greater than 2 and smaller than 3". Or we could be more precise and say

$$\frac{30}{13} = 2 + \frac{4}{13}.$$

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The integer 2 is called the *integer part* of $\frac{30}{13}$ and the rational number $\frac{4}{13}$ is the *remainder.* Another way of writing the above equation would be

$$\frac{30}{13} = 2 + \frac{1}{(\frac{13}{4})}$$

and although it may seem strange at first sight, I prefer this way. The reason is that now we can continue our procedure with the fraction $\frac{13}{4}$. This number is bigger than 3 and smaller than 4. In fact, we have

$$\frac{13}{4} = 3 + \frac{1}{4},$$

which means of course that

$$\frac{30}{13} = 2 + \cfrac{1}{3 + \cfrac{1}{4}}. \tag{1}$$

Note that now the numerator of the remainder $\frac{1}{4}$ is equal to 1. Therefore, if we repeat the same procedure again, we would replace a fraction $\frac{1}{4}$ by 1 divided by its inverse $\frac{4}{1}$, but this would not change anything, since $\frac{4}{1} = 4$, obviously. Thus we can stop our construction as soon as the numerator of the remainder is 1.

The expression in (1) is called the *continued fraction expansion* of $\frac{30}{13}$. Since all numerators are equal to 1, we will usually just write $[2, 3, 4]$ for the right hand side of (1). There is nothing special here about the integers 30 and 13; we can compute such a continued fraction expansion for any rational number $\frac{p}{q}$, although we might need more than just two steps.

Conversely, given a sequence of positive integers we can certainly compute the rational number of the continued fraction determined by it. If we take the sequence $[3, 2, 2]$ we find

$$[3, 2, 2] = 3 + \cfrac{1}{2 + \cfrac{1}{2}} = 3 + \cfrac{1}{\cfrac{5}{2}} = 3 + \frac{2}{5} = \frac{17}{5}.$$

Let's write down a formal definition.

**Definition 1.1.** *A* finite positive continued fraction *is an expression of the form*

$$[a_1, a_2, a_3 \ldots, a_n] = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$$

*where* $a_1, a_2, a_3 \ldots, a_n$ *are positive integers. Similarly, an* infinite positive continued fraction *is an expression of the form*

$$[a_1, a_2, a_3 \ldots] = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots}}}$$

*where we have an infinite sequence of positive integers* $a_1, a_2, a_3, \ldots$.

If we were to compute a positive continued fraction $[a_1, a_2, a_3 \ldots, a_n]$ whose last entry $a_n$ is equal to 1, then in the last step we would have $a_{n-1} + \frac{1}{1}$ which is equal to $a_{n-1} + 1$. This shows that $[a_1, a_2, \ldots, a_{n-1}, 1] = [a_1, a_2, \ldots, a_{n-1} + 1]$. Thus it suffices to consider continued fractions whose last entry is at least 2. The following classical result, which can be found (for instance) in [5], shows that this is the only ambiguity for finite continued fractions.

**Theorem 1.2.** *There is a bijection between the set* $\mathbb{Q}_{>1}$ *of rational numbers that are greater than 1 and the set of finite positive continued fractions whose last coefficient is at least 2.*

For infinite continued fractions we have the following classical result, which can also be found in [5].

**Theorem 1.3.** *There is a bijection between the set* $\mathbb{R}_{>1} \setminus \mathbb{Q}_{>1}$ *of real numbers greater than 1 that are not rational and the set of infinite positive continued fractions.*

Continued fractions have been studied, at least implicitly, since antiquity. Evidence of the continued fraction expansion can be found in the approximation of $\pi$ given by Archimedes, and in the division algorithm of Euclid, which we will return to in Section 4. The modern theory started with Euler in the 18th century. An introduction to the topic can be found in most books on number theory, for example in Chapter 10 of [5].

In some cases, the intrinsic beauty of a real number really becomes apparent in its continued fraction expansion. For example the number known as the *golden ratio* which is equal to $\frac{1+\sqrt{5}}{2} \approx 1.618$ has the simplest possible infinite continued fraction expansion

$$\frac{1+\sqrt{5}}{2} = [1, 1, 1, 1, \ldots].$$

Sometimes certain patterns show up in the continued fraction. A famous example is the base of the natural logarithm $e \approx 2.718$ (Euler's number). It has the following infinite continued fraction expansion:

$$e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1, 14, 1, 1, 16, 1, 1, 18, \ldots].$$

## 2 Snake graphs

We now change our point of view and give a combinatorial interpretation of continued fractions. We will use the word *tile* for a square in the plane whose sides are only horizontal and vertical. To describe directions, we will use the words east and west (for left and right), and north and south (for up and down).

We are now going to build snake graphs out of tiles.[2] Take a certain number of tiles, let's say $d$ tiles, and start by laying down a first tile. Then place a second tile either to the east of the first tile or to the north. You have the choice here, and one way or the other will not produce the same snake graph. Then place a third tile either to the east or to the north of the second tile. Again you must choose. Continue this way until you have used all of the $d$ tiles. An example is shown in Figure 1. We will use the letter $\mathcal{G}$ to denote the snake graph.

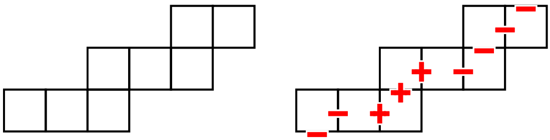

Figure 1: A snake graph (left) and the same snake graph with its sign sequence (right)

Each tile has 4 vertices and 4 edges. A snake graph with 2 tiles has 6 vertices and 7 edges, since the two tiles share two vertices and one edge. Each additional tile produces 2 new vertices and 3 new edges, and therefore a snake graph with $d$ tiles has $2d + 2$ vertices and $3d + 1$ edges. The $d - 1$ edges that are shared by two consecutive tiles are called *interior edges*, and we denote them $e_1, e_2, \ldots, e_{d-1}$. We order them such that the interior edge $e_i$ lies between the $i$th and $(i + 1)$th tile as they are laid down.

To remember which snake graph you have constructed it is convenient to use a *sign function* $f$. This is a map $f$ from the set of edges of $\mathcal{G}$ to $\{+1, -1\}$ such that on every tile in $\mathcal{G}$ the north and the west edge have the same sign, the south and the east edge have the same sign and the sign on the north edge is opposite to the sign on the south edge. See Figure 1 for an example which shows the sign on the interior edges.

A snake graph $\mathcal{G}$ is determined by the number of tiles it is made from and the values of the sign function $f$ on its interior edges. In addition, we will let $e_0$ be the south edge of the first tile and we choose one of the two edges, north

[2] Snake graphs first appeared in the context of cluster algebras in [6, 7, 8] and were then studied in a conceptual way in [1, 2, 3]. We follow the presentation in [4].

or east, of the last tile and denote it by $e_d$. Then we obtain a sign sequence

$$(f(e_0), f(e_1), \ldots, f(e_{d-1}), f(e_d)). \tag{2}$$

This sequence uniquely determines the snake graph and the choice of either the north or east edge of the last tile as $e_d$. We encourage you to choose for yourself a sign sequence and draw the associated snake graph.

## 3 The snake graph of a continued fraction

Now let $[a_1, a_2, \ldots, a_n]$ be a positive finite continued fraction, and let $d = a_1 + a_2 + \cdots + a_n - 1$. Consider the following sign sequence

$$(\underbrace{-,\ldots,-}_{a_1},\ \underbrace{+,\ldots,+}_{a_2},\ \underbrace{-,\ldots,-}_{a_3},\ \underset{\ldots}{\ldots},\ \underbrace{\pm,\ldots,\pm}_{a_n}), \tag{3}$$

where each integer $a_i$ corresponds to a maximal subsequence of constant sign.

**Definition 3.1.** *The snake graph $\mathcal{G}[a_1, a_2, \ldots, a_n]$ of the positive continued fraction $[a_1, a_2, \ldots, a_n]$ is the snake graph with d tiles determined by the sign sequence (3).*

For example, the snake graph in Figure 1 corresponds to the continued fraction $[2, 3, 4]$.

In order to really establish a relation to continued fractions, we need the notion of perfect matchings. A *perfect matching* of a snake graph $\mathcal{G}$ is a subset $P$ of the set of edges of $\mathcal{G}$ such that every vertex of $\mathcal{G}$ is covered by exactly one edge in $P$. For example, the snake graph $\mathcal{G}[2, 2]$ has precisely 5 perfect matchings as shown in Figure 2.

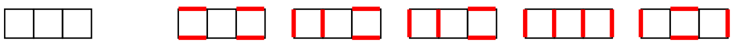

Figure 2: The snake graph $\mathcal{G}[2, 2]$ (left), and its 5 perfect matchings (right).

The following result has been proved very recently, in 2016 [4].

**Theorem 3.2.** *If $m(\mathcal{G})$ denotes the number of perfect matchings of $\mathcal{G}$ then*

$$[a_1, a_2, \ldots, a_n] = \frac{m(\mathcal{G}[a_1, \ldots, a_n])}{m(\mathcal{G}[a_2, \ldots, a_n])}$$

*and this fraction is reduced (that is, the numerator and denominator do not have common divisors).*

This theorem gives a combinatorial realization of continued fractions. This means that the numerator and the denominator of the continued fraction are realized as the number of elements of sets that are constructed from the continued fraction. In other words, these numbers count something.

**Example 3.3.** *The snake graph $\mathcal{G} = \mathcal{G}[1, 1, \ldots, 1]$ is the straight snake graph with $n-1$ tiles and its number of perfect matchings is the $n+1$-st Fibonacci number. The first few values are given in the table below.*

| $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $m(\mathcal{G})$ | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 |

# 4 Division algorithm

Let's go back to our example of the continued fraction $[2, 3, 4] = \frac{30}{13}$. Its snake graph $\mathcal{G}[2, 3, 4]$ is given in Figure 1. To compute the continued fraction starting from the rational number $\frac{30}{13}$, we used the following division algorithm.

$$\begin{aligned} 30 &= 2 \cdot 13 + 4 \\ 13 &= 3 \cdot 4 + 1 \\ 4 &= 4 \cdot 1. \end{aligned}$$

We can see this division algorithm on the level of the snake graph $\mathcal{G}[2, 3, 4]$ and its subgraphs as illustrated in Figure 3. In this figure, the numbers in the tiles represent the number of perfect matchings of the snake graph from that tile onward to the end. Thus the number 30 in the first tile of $\mathcal{G}[2, 3, 4]$ means that $\mathcal{G}[2, 3, 4]$ has exactly 30 perfect matchings. The number 17 in the second tile means that the snake graph obtained from $\mathcal{G}[2, 3, 4]$ by removing the first tile has 17 perfect matchings, and so on.

The first line in the figure counts the number of perfect matchings of $\mathcal{G}[2, 3, 4]$ as follows. Start by choosing one of the two perfect matchings of the first tile. How many ways are there to complete this choice to a perfect matching of the whole snake graph $\mathcal{G}[2, 3, 4]$? The only restriction is that we cannot use the two horizontal edges of the second tile, since it would clash with the edges already chosen on the first tile. Thus the number of ways to complete a matching of the first tile to a matching of the whole graph is exactly the number of perfect matchings of the graph consisting of the last 6 tiles of $\mathcal{G}[2, 3, 4]$ shown in the first row of the figure. Note that this snake graph is exactly $\mathcal{G}[3, 4]$. If you believe for now that this graph has 13 perfect matchings, this accounts for $2 \cdot 13 = 26$ perfect matchings. But there are still a few more perfect matchings of $\mathcal{G}[2, 3, 4]$, namely those that do not restrict to a perfect matching of the first tile. Such a perfect matching must contain the two horizontal edges of the second tile, and therefore must contain the west edge of the first tile, the east edge of the third tile, and the north edge of the fourth tile. Thus it is completely determined except for the last three tiles, which means that there are exactly 4 ways to complete it. Altogether this gives the equation $30 = 2 \cdot 13 + 4$ as shown in the first row of Figure 3.

The second equation of the algorithm is shown in the second row of the Figure. We want to count the number of perfect matchings of the snake graph $\mathcal{G}[3,4]$. First we choose any of the 3 perfect matchings of the subgraph consisting of the first 2 tiles (try to see for yourself why we don't start here with only the first tile). The number of ways to complete it to a perfect matching of $\mathcal{G}[3,4]$ is then precisely the number of perfect matchings of the graph given by the last 3 tiles. Note that this graph is equal to $\mathcal{G}[4]$. Finally, we need to count the perfect matchings that do not restrict to a perfect matching of the first two tiles. Such a perfect matching must use the two horizontal edges of the third tile (the tile with label 5). Completing it in the front of the snake graph, we need the west edge of the second tile and the south edge of the first. At the other end the completion also is uniquely determined. We need the west edge on the fourth tile, the north edge on the fifth tile and the west edge on the last tile. Thus the total number of perfect matchings of $\mathcal{G}[3,4]$ is $3 \cdot 4 + 1 = 13$.

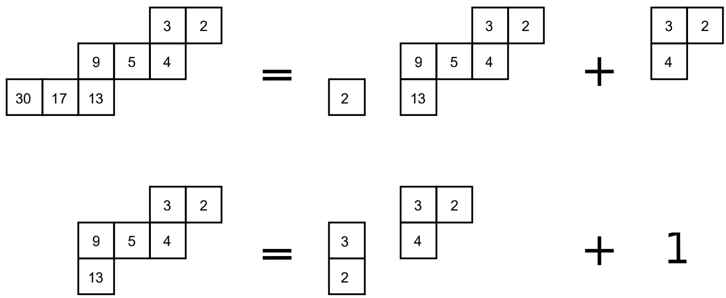


Figure 3: The division algorithm in terms of perfect matchings of snake graphs

## 5 An application

To give just one illustration of why this combinatorial interpretation of continued fractions is interesting let us prove the following theorem.

**Theorem 5.1.** *The continued fractions* $[a_1, \ldots, a_n]$ *and* $[a_n, \ldots, a_1]$ *have the same numerator.*

*Proof.* The numerator of $[a_1, \ldots, a_n]$ is the number of perfect matchings of $\mathcal{G}[a_1, \ldots, a_n]$, and the numerator of $[a_n, \ldots, a_1]$ is the number of perfect matchings of $\mathcal{G}[a_n, \ldots, a_1]$. But these two snake graphs are obtained from each other by a rotation of 180 degrees, so they have the same number of perfect matchings. □

This theorem has been known for a long time and can be proved with a little effort and without much difficulty using the recursive definition of the continued fraction and the division algorithm. However, one has to go through this effort to obtain a proof. The beauty of the snake graph approach is that the proof becomes completely obvious.

## Image credits

All images created by the author.

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