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Rotating needles, vibrating strings, and Fourier summation

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We give a brief survey of the connection between seemingly unrelated problems such as sets in the plane containing lines pointing in many directions, vibrating strings and drum heads, and a classical problem from Fourier analysis.

In 1917, the mathematician Sōichi Kakeya (1886–1947) posed the following question [1]: Imagine that an infinitely thin needle covered in ink has been placed on a sheet of paper so that the sharp end points straight up. Your job is to slide and rotate (but not lift!) the needle so that it points straight down; the goal is to leave the least possible amount of ink on the paper (that is, the inked region should have the smallest possible area). How good of a job can you do?

Rather surprisingly, two years later Abram Samoilovitch Besicovitch (1891–1970) discovered while working on a seemingly unrelated problem that the area stained by the ink can be arbitrarily small [2] – in other words, for any positive number r, there is a sequence of rotations and translations so that the ink left on the paper has area at most r. The pattern made by the ink on the paper has a curious property: it has very little area, but it contains a line segment pointing in each direction. Sets in the plane that contain a line segment pointing in each direction are now called *Besicovitch sets*, and they have unexpected connections to many fields of mathematics. One of the most surprising connections was found by Charles Fefferman in the 1970s [3]. Before we can discuss Fefferman's work, however, we need to go back by about 200 years, when the problem of vibrating strings was first given a proper mathematical description.

A violin string can vibrate in several ways, which are called *vibrational modes*. First, there is the fundamental frequency (also known as first harmonic), where the entire string vibrates back and forth – just the endpoints are fixed. Then there is the second harmonic, where the endpoints and midpoint of the string remain stationary while the two segments between them can move freely; the third harmonic, where there are two stationary points (respectively a third and two thirds away from the endpoints); and so on. For simplicity, we will suppose that the string has length π. We will use the variable x to represent the distance along the violin string from its first end; it will thus take values between 0 and π. At maximum displacement, the position of each point of the violin string vibrating at its first harmonic would be represented by a function of our variable x, given by the expression $f_1(x) = a_1 \sin(x)$, where a_1 is a real number. This number is called *amplitude*, it is the value of the highest point the string can reach in the direction orthogonal to the one given by the string at rest. Notice that for this first harmonic $a_1 \sin(x) = 0$ if $x = 0$ or $x = \pi$, and $a_1 \sin(x) = a_1$ if $x = \pi/2$, namely, the endpoints are at rest and the highest point is reached at the center of the string. In general, the position of a violin string vibrating at its n-th harmonic would be represented at its maximum displacement by the equation $f_n(x) = a_n \sin(nx)$.

A violin string can also vibrate in a superposition of several of the different vibrational modes described above. For example, it can vibrate in a combination of the fundamental frequency and the third harmonic. This would be represented by the equation $f(x) = a_1 \sin(x) + a_3 \sin(3x)$, which is simply the sum of the two functions characterizing these two harmonics. Such superpositions are the only ways a violin string can vibrate, that is, at maximum displacement, the position of any violin string can be represented by a sum of harmonics with an expression of the form $f(x) = \sum_n a_n \sin(nx)$.

However, a sufficiently skilled violinist could make a string vibrate in any pattern she wishes. This means that it should be possible for a violin string at maximum displacement to resemble any function $f(x)$, provided that the two ends of the string are fixed (that is, $f(0) = 0$ and $f(\pi) = 0$), and that the function $f(x)$ is differentiable (namely, the string doesn't have any kinks). This created a seeming paradox: a violin string at maximum displacement can resemble any differentiable function whatsoever, and yet that vibration can also be thought of as a superposition of vibrational modes.

One of the first mathematicians to help resolve this conundrum was Jean Baptiste Joseph Fourier (1768–1830). He showed that no matter how you cause a violin string to vibrate, the position of the string at maximum displacement can be expressed as a combination vibrational modes. In particular, any differentiable function $f(x)$ (defined for all x between 0 and π) satisfying $f(0) = 0$

and $f(\pi) = 0$ can be written as a sum of harmonics:

$$f(x) = \sum_n a_n \sin(nx), \tag{1}$$

where the numbers a_n are called the coefficients of the Fourier sine series of f, and are given by

$$a_n = \frac{2}{\pi} \int_0^\pi f(x) \sin(nx). \tag{2}$$

The catch is that the number of modes required (that is, the number of terms in the sum (1)) might be infinite. At the time, Fourier and his contemporaries did not have a mathematically rigorous way of dealing with this type of infinite summation.

One way to make sense of an infinite sum like (1) is to choose a large integer N and to only sum the terms $a_n \sin(nx)$ from (1) with $n \leq N$, that is we could consider *partial sums* of the form

$$\sum_{n=1}^N a_n \sin(nx). \tag{3}$$

It turns out that as N becomes larger, this partial sum becomes an increasingly good approximation for the original function f. Mathematically, we say that the partial sum converges to f. This is true if the original function f is differentiable, and it can sometimes remain true even if the function $f(x)$ isn't differentiable, as long as the function $f(x)$ is reasonably well behaved. These ideas launched a vast and vibrant area of mathematics called Fourier analysis, which has connections to areas as diverse as physics, electrical engineering, and signal processing. Though some subtle questions are still being investigated, Fourier analysis in one dimension is now relatively well understood.

Sums of the form (1) are useful for analyzing functions that are defined on an interval $[0, \pi]$, or equivalently, functions that are defined for all real numbers but that are 2π-*periodic*, which means that $f(x) = f(x + 2\pi)$ for all $x \in \mathbb{R}$[1]. Fourier analysis has also developed tools for analyzing functions that are defined for all real numbers and are not necessarily periodic. One such was of representing a function of this form is

$$g(x) = \int e^{-ixs} h(s) ds, \tag{4}$$

where for each real number s, the number $h(s)$ is given by

$$h(s) = \frac{1}{2\pi} \int e^{ixs} g(x) dx.$$

[1] As a technical aside, the sum (1) can only express functions that are 2π-periodic, that vanish at zero, and that are *odd*, which means that $f(x) = -f(-x)$ for all real numbers x.

The function h is called the *Fourier transform* of g. The values of this function are the equivalent of the Fourier coefficients for the summation (1).

In analogy with the partial sums (3), we could consider the partial integral

$$S_N g(x) = \int_{-N}^{N} e^{-ixs} h(s) ds, \tag{5}$$

and we could ask whether the function $S_N g(x)$ is a good approximation for $g(x)$ when N is large. There are several different ways that this question can be phrased, and these different questions sometimes yield different answers. One way of making this question precise is to measure the so-called L^p *error*

$$\int |g(x) - S_N g(x)|^p dx,$$

for some real number $p \geq 1$, and ask whether this error becomes smaller as N grows. If the original function g is well behaved, then the integral (5) does an increasingly good job at approximating the function g, and the L^p error becomes smaller.

What happens if we move from functions of one variable to functions of two variables? If a function of one variable represents the maximum displacement of a vibrating string, then a function of two variables represents the maximum displacement of a vibrating drum head. If $g(x, y)$ is a function of two variables, then the analogue of (4) would be

$$g(x, y) = \iint e^{-i(xs+yt)} h(s, t) ds dt, \tag{6}$$

where for each pair of real numbers s and t, the number $h(s, t)$ is given by

$$h(s, t) = \frac{1}{(2\pi)^2} \iint e^{i(xs+yt)} g(x, y) dx dy.$$

Again, we might expect that as N becomes larger, partial integrals like

$$S_N^{\square} g(x, y) = \int_{-N}^{N} \int_{-N}^{N} e^{-i(xs+yt)} h(s, t) ds dt \tag{7}$$

or

$$S_N^{\bigcirc} g(x, y) = \iint_{B(N)} e^{-i(xs+yt)} h(s, t) ds dt, \tag{8}$$

where $B(N) = \{(s, t) \in \mathbb{R}^2 \colon \sqrt{s^2 + t^2} \leq N\}$, should be good approximations for $g(x, y)$. The function $S_N^{\square} g(x, y)$ is labeled with a square because we are integrating over the box $\{(s, t) \in \mathbb{R}^2 \colon -N \leq s \leq N,\ -N \leq t \leq N\}$.

The sum $S_N^{\bigcirc} g(x,y)$ is instead labeled with a circle since we are integrating over the disk $B(N)$.

It turns out that $S_N^{\square} g$ is a good approximation for g, in the sense that the L^p error $\int\int |g(x,y) - S_N^{\square} g(x,y)|^p dxdy$ becomes small as N becomes large. It was generally assumed that the approximation $S_N^{\bigcirc} g$ should behave similarly. In 1971, however, Charles Fefferman showed that this was not the case: for certain functions g, the approximation $S_N^{\bigcirc} g$ could behave very differently from $S_N^{\square} g$, and therefore from g. The function g that Fefferman constructed was closely related to the Besicovitch sets described above. Specifically, he considered a Besicovitch set E in the plane that had very small area, and he defined g to be the function that takes the value $g(x,y) = 1$ when $(x,y) \in E$ and $g(x,y) = 0$ when $(x,y) \notin E$. The function g therefore describes a set that has small area and contains line segments pointing in every direction, while $S_N^{\bigcirc} g$ describes a very different set: the operation of applying $S_N^{\bigcirc}$ shifts each of the lines, so that they no longer overlap. Thus the functions g and $S_N^{\bigcirc} g$ look very different. This unexpected discovery connected the geometric world of Besicovitch sets to the analytic world of Fourier analysis; and by doing so, it created a new area of mathematics that seeks to explore these connections.

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References

[1] S. Kakeya, *Some problems on maximum and minimum regarding ovals*, Tohoku science reports **6** (1917), 71–88.

[2] A. Besicovitch, *Sur deux questions d'intégrabilite des fonctions*, Journal de la société physico-mathématique **2** (1919), 105–123.

[3] C. Fefferman, *The multiplier problem for the ball*, Annals of mathematics **94**, 2 (1971), 330–336.

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