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Ultrafilter methods in combinatorics

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Given a set X, ultrafilters determine which subsets of X should be considered as *large*. We illustrate the use of ultrafilter methods in combinatorics by discussing two cornerstone results in Ramsey theory, namely Ramsey's theorem itself and Hindman's theorem. We then present a recent result in combinatorial number theory that verifies a conjecture of Erdős known as the "$B + C$ conjecture".

1 Ramsey's theorem

The *(infinite) pigeonhole principle* asserts that if we color every element of an infinite set, let us say the set of natural numbers $\mathbb{N}$, with one of two colors, say red or blue, then some infinite subset X of $\mathbb{N}$ is *monochromatic*, that is every element of X has the same color. The proof basically amounts to saying that the union of two finite sets is finite. Ramsey's theorem is a significant generalization of the pigeonhole principle.

Consider *pairs* of natural numbers, that is, sets of the type $\{m, n\}$ with m, $n \in \mathbb{N}$ and $m \neq n$. By coloring a pair, we mean that we assign to this set a specific color.

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Theorem 1 (Ramsey's theorem) [2] *Suppose that one colors all pairs of natural numbers with two colors, say red or blue. Then there is an infinite subset X of $\mathbb{N}$ that is monochromatic, in the sense that all possible pairs of elements from X receive the same color.*

In the remainder of this section, we give an idea of the proof of this theorem by using the fictitious quantifier $\forall^*$. Intuitively, the quantifier $\forall^* x$ should be thought of as saying "for most elements" x or "for a large number of elements" x. In the next section, we show that such a quantifier can actually be shown to exist using the notion of ultrafilters.

Represent a pair of natural numbers by an ordered pair (m, n) with $m < n$ and fix a coloring c on such pairs of numbers. We write $R(m, n)$ if c assigns to the pair (m, n) the color red, and $B(m, n)$ if the color blue is assigned.

We seek to construct a sequence $a_1 < a_2 < a_3 < \dots$ of natural numbers such that either $R(a_m, a_n)$ holds for all $m < n$, or $B(a_m, a_n)$ holds for all $m < n$. To know which color is more likely, it would be nice if either

$$\forall^* x \, \forall^* y \, R(x, y) \quad \text{or} \quad \forall^* x \, \forall^* y \, B(x, y)$$

holds, meaning that, in the first case, many x have the property that there are many y for which $R(x, y)$ holds, and, in the second, that many x have the property that there are many y for which $B(x, y)$ holds. This is indeed the case when our quantifier satisfies the following property:

> **(Part)** For all subsets $X, Y \subseteq \mathbb{N}$ such that $\mathbb{N} = X \cup Y$, we have either $\forall^* x \, (x \in X)$ or $\forall^* x \, (x \in Y)$.

Note for example that the quantifier "for all but finitely many x", which is the quantifier often seen in analysis (for example in the definition of limit), does not satisfy requirement (Part). Indeed, this can be easily seen if you take X as the set of even numbers and Y as the set of odd numbers. Assuming (Part) for our quantifier $\forall^*$, we now have a color, let us say red, such that $\forall^* x \, \forall^* y \, R(x, y)$ holds. We now fix $a_1 \in \mathbb{N}$ such that $\forall^* y \, R(a_1, y)$. Actually, in order to guarantee the existence of a_1, we have to assume the following property:

> **(Non-$\emptyset$)** If $X \subseteq \mathbb{N}$ is such that $\forall^* x \, (x \in X)$ holds, then $X \neq \emptyset$.

In other words, if something happens for "many" elements x, then it should happen for at least one x! How do we proceed? We know that for many x,

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[2] Technically this is known as the infinite Ramsey theorem for pairs with two colorings. An inductive argument is needed to go from two colors to an arbitrary finite number of colors. One can also replace pairs by triples, quadruples, and so on, which makes the proof harder only in notation.

there are many y for which $R(x, y)$ holds and that there are many y such that $R(a_1, y)$ holds. We now desire an a_2 that satisfies both of these properties. In other words, a_2 should lie in the intersection of these two large sets, motivating the following property:

(Int) If $\forall^* x\,(x \in X)$ and $\forall^* x\,(x \in Y)$, then $\forall^* x\,(x \in X \cap Y)$.

However, we also want a_2 to be larger than a_1, which could in theory be impossible if a large set was finite. We thus consider the following strengthening of (Non-$\emptyset$):

(Inf) If $\forall^* x\,(x \in X)$, then X is infinite.

Using (Int) and (Inf), we can thus choose $a_2 > a_1$ such that $R(a_1, a_2)$ and $\forall^* y\, R(a_2, y)$. We now have everything we need to continue our construction. We find $a_3 > a_2$ such that $R(a_1, a_3)$, $R(a_2, a_3)$ and $\forall^* y\, R(a_3, y)$. This uses the fact that the intersection of three sets is large, which follows from applying (Int) twice. The rest of the construction proceeds in a similar manner.

This concludes the proof of Ramsey's theorem, *assuming* the existence of a quantifier $\forall^* x$ satisfying (Part), (Int), and (Inf). We show the existence of such a quantifier in the next section, by discussing the notion of ultrafilters.

2 Basic facts on ultrafilters

The definition of an ultrafilter is often stated in a slightly different manner than the terminology used in the previous section. First, we introduce the notion of filters.

Definition 2 *A* filter[3] *on* $\mathbb{N}$ *is a collection* $\mathcal{F}$ *of subsets of* $\mathbb{N}$ *satisfying the following three properties:*

1. *The empty set* $\emptyset$ *does not belong to* $\mathcal{F}$ *while* $\mathbb{N}$ *itself does belong to* $\mathcal{F}$.
2. *If* A *belongs to* $\mathcal{F}$ *and* $A \subseteq B$, *then* B *also belongs to* $\mathcal{F}$.
3. *If* A *and* B *both belong to* $\mathcal{F}$, *then* $A \cap B$ *also belongs to* $\mathcal{F}$.

If one thinks of a coffee filter, used to catch the "large" coffee grinds, a filter on $\mathbb{N}$ catches the "large" subsets of $\mathbb{N}$. Here large can mean different things depending on different filters. The first property says that the empty set should not be large, which corresponds to requirement (Non-$\emptyset$) from the previous section, while the entire set itself is large. The second property says

[3] Technically we describe the notion of a *proper* filter. The improper filter on $\mathbb{N}$ is simply the collection of all subsets of $\mathbb{N}$. We do not allow this as a filter in this snapshot.

that if A is large and B is even larger, then B is also large. The final property corresponds exactly to requirement (Int) from the previous section.

A prominent example of a filter is the following.

Example 3 *The* Frechet filter *on* $\mathbb{N}$ *consists of those subsets* A *of* $\mathbb{N}$ *for which the complement* A^c*, that is, the set of numbers not belonging to* A*, is finite.*

Often one wants to know if a set of elements satisfying a given property is, or is not, an element of a filter $\mathcal{F}$. A useful notation is provided by the filter quantifier $\forall^{\mathcal{F}}$, which means that the set of elements which makes the property true belongs to $\mathcal{F}$. For example, for a set A, the expression $\forall^{\mathcal{F}} x\,(x \in A)$ means that the set of x which are contained in A is in the filter $\mathcal{F}$. In general, such a quantifier does not satisfy property (Part) from the previous section. For instance, if $\mathcal{F}$ is the Frechet filter, then partitioning the numbers into the set of even numbers and the set of odd numbers witnesses that $\forall^{\mathcal{F}}$ fails requirement (Part). This is why we need to consider ultrafilters.

Definition 4 *A filter* $\mathcal{F}$ *on* $\mathbb{N}$ *is called an* ultrafilter *if it also satisfies:*

4. For every $A \subseteq \mathbb{N}$*, either* A *belongs to* $\mathcal{F}$ *or* A^c *belongs to* $\mathcal{F}$*.*

Thus, the Frechet filter is not an ultrafilter. We often use the letters $\mathcal{U}$ and $\mathcal{V}$ to denote ultrafilters. One can show that, for an ultrafilter $\mathcal{U}$, $\forall^{\mathcal{U}}$ satisfies the properties (Part) and (Int). But what about (Inf)? Unfortunately, this is not always the case for (Inf), as the next example shows.

Example 5 *Fix a number* n *in* $\mathbb{N}$*. Let* $\mathcal{U}_n$ *consist of all subsets of* $\mathbb{N}$ *which contain* n*. Then* $\mathcal{U}_n$ *is an ultrafilter on* $\mathbb{N}$*, called the* principal ultrafilter generated by *n.*

Since the set $\{n\}$ consisting just of n belongs to $\mathcal{U}_n$, the quantifier $\forall^{\mathcal{U}_n}$ fails the axiom (Inf) miserably! However, this is the only case in which an ultrafilter can fail (Inf). It turns out that if $\mathcal{U}$ is a *nonprincipal* ultrafilter, that is $\mathcal{U} \neq \mathcal{U}_n$ for any $n \in \mathbb{N}$, then $\forall^{\mathcal{U}}$ does satisfy (Inf) as well.[4]

To summarize, if $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{N}$, then the quantifier $\forall^{\mathcal{U}}$ satisfies the axioms (Part), (Int), and (Inf) from the previous section. However, the question remains: do nonprincipal ultrafilters on $\mathbb{N}$ exist? Thankfully, they do. The argument goes by first showing that an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is a “maximal” filter, in the sense that there is no filter that extends $\mathcal{U}$, and then by using *Zorn's Lemma,* which ensures the existence of such a maximal filter. In fact, one shows that every filter is contained in an ultrafilter, so that there exist many, many nonprincipal ultrafilters.

[4] This is a nice exercise for the reader to work out.

The existence of a nonprincipal ultrafilter satisfying (Part), (Int) and (Inf) concludes the proof of Ramsey's theorem by allowing the construction of the sequence $a_1, a_2, \ldots$ from the previous section.

3 Hindman's theorem and the semigroup $\beta\mathbb{N}$

Another interesting and nontrivial extension of the pigeonhole principle is *Hindman's theorem*. Consider $X \subseteq \mathbb{N}$ and let $\mathrm{FS}(X)$ denote the set of all finite sums of distinct elements of X. In other words, if $X = \{x_1, x_2, \ldots\}$, then $\mathrm{FS}(X)$ consists of the elements of X itself as well as elements such as $x_1 + x_2$ and $x_3 + x_6 + x_{15} + x_{1000}$.

Theorem 6 (Hindman's Theorem) *If one colors all elements of* $\mathbb{N}$ *with the colors red and blue, then there is an infinite subset* X *of* $\mathbb{N}$ *such that all elements of* $\mathrm{FS}(X)$ *receive the same color.*

Hindman's original proof was purely combinatorial and very tricky.[5] A simpler proof is given by considering ultrafilters and defining a sequence in a manner similar to the proof of Ramsey's theorem. More precisely, one constructs an infinite sequence by keeping a large number of options open for future choices of sequence elements. In this case, it is important that large sets have the property that a large number of shifts of the set remain large. Given $A \subseteq \mathbb{N}$ and $a \in \mathbb{N}$, a shift of A by a is defined by

$$A - a := \{b \in \mathbb{N} : \; a + b \in A\}.$$

In other words, we shift A to the left by a units and then throw away any negative numbers that might arise. Now, given an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and $A \subseteq \mathbb{N}$, we set

$$A - \mathcal{U} := \{a \in \mathbb{N} : \; A - a \in \mathcal{U}\}.$$

Namely, $A - \mathcal{U}$ contains all those a for which the shift of A by a is large in the sense of $\mathcal{U}$. Note that if $\mathcal{U}$ is the principal ultrafilter $\mathcal{U}_b$ for some $b \in \mathbb{N}$, then $a \in A - \mathcal{U}_b$ means $A - a \in \mathcal{U}_b$, implying $b \in A - a$, that is $a \in A - b$. This means that this notion of ultrafilter shift generalizes the above notion of shifting a set by a number. The key definition for the ultrafilter proof of Hindman's theorem is the following.

Definition 7 *An ultrafilter* $\mathcal{U}$ *on* $\mathbb{N}$ *is called* idempotent *if, for every* $A \in \mathcal{U}$, *one has* $A - \mathcal{U} \in \mathcal{U}$.

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[5] Hindman himself suggested that asking graduate students to read the original proof could be viewed as a form of torture!

Alternatively stated, $\mathcal{U}$ is idempotent if, whenever A is large, then a large number of shifts of A are also large. The terminology idempotent can be explained as follows. Let $\beta\mathbb{N}$ denote the set of all ultrafilters on $\mathbb{N}$.[6] One can define an addition operation $\oplus$ on $\beta\mathbb{N}$ by declaring, for ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on $\mathbb{N}$ and $A \subseteq \mathbb{N}$, that $A \in \mathcal{U} \oplus \mathcal{V}$ if and only if $A - \mathcal{V} \in \mathcal{U}$.[7] Thus, A is large in the sense of $\mathcal{U} \oplus \mathcal{V}$ if and only if a $\mathcal{U}$-large number of shifts of A are $\mathcal{V}$-large. The operation $\oplus$ on $\beta\mathbb{N}$ is a semigroup operation, meaning that $(\mathcal{U} \oplus \mathcal{V}) \oplus \mathcal{W} = \mathcal{U} \oplus (\mathcal{V} \oplus \mathcal{W})$. Note indeed that the operation agrees with the usual addition on $\mathbb{N}$ in case $\mathcal{U}$ and $\mathcal{V}$ are both principal. With this terminology, saying that an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ is idempotent is equivalent to say that $\mathcal{U} \oplus \mathcal{U} = \mathcal{U}$, which is exactly what it means for an element in a semigroup to be idempotent. Note also that all idempotent ultrafilters are nonprincipal.[8]

A key ingredient to prove Hindman's theorem is the following proposition.

Proposition 8 *Suppose $\mathcal{U}$ is an idempotent ultrafilter. Then for every $A \in \mathcal{U}$, there is an infinite set $X \subseteq \mathbb{N}$ such that* $\mathrm{FS}(X) \subseteq A$.

We briefly discuss the proof of this proposition by showing how to construct the first elements of the set $X = \{a_1, a_2, \ldots\}$. Since $\mathcal{U}$ is idempotent and $A \in \mathcal{U}$, we have that $A - \mathcal{U} \in \mathcal{U}$, hence $A \cap (A - \mathcal{U}) \in \mathcal{U}$. Fix $a_1 \in A \cap (A - \mathcal{U})$. Since $a_1 \in A - \mathcal{U}$, we have $A - a_1 \in \mathcal{U}$. This implies $(A - a_1) - \mathcal{U} \in \mathcal{U}$, hence $A \cap (A - \mathcal{U}) \cap (A - a_1) \cap (A - a_1 - \mathcal{U}) \in \mathcal{U}$. Since idempotent ultrafilters are nonprincipal, there exists $a_2 > a_1$ such that $a_2 \in A \cap (A - \mathcal{U}) \cap (A - a_1) \cap (A - a_1 - \mathcal{U})$. At this point we have that $a_1, a_2, a_1 + a_2 \in A$, and we are on our way to construct our desired infinite set X. Let us construct one more element of the sequence. Since $\mathcal{U}$ is idempotent and a_2 is chosen in such a way that $A - a_2, A - a_1 - a_2 \in \mathcal{U}$, we have that $(A - a_2) - \mathcal{U} \in \mathcal{U}$ and $(A - a_1 - a_2) - \mathcal{U} \in \mathcal{U}$, hence, we can find $a_3 > a_2$ such that $a_3 \in A \cap (A - \mathcal{U}) \cap (A - a_1) \cap (A - a_1 - \mathcal{U}) \cap (A - a_2) \cap (A - a_2 - \mathcal{U}) \cap (A - a_1 - a_2) \cap (A - a_1 - a_2 - \mathcal{U})$, since $\mathcal{U}$ is nonprincipal. Now $a_3, a_1 + a_3, a_2 + a_3, a_1 + a_2 + a_3 \in A$. The proof continues in this manner and a clever choice of notation makes a complete inductive proof fairly clean to write down.

Recall that to prove Hindman's theorem we need to find an infinite set $X = \{x_1, x_2, \ldots\}$ such that all elements of $FS(X)$ receive the same color. By Proposition 8, we know that the existence of an idempotent ultrafilter implies that any large set contains $FS(X)$ for some subset $X \subseteq \mathbb{N}$. We are now ready for the proof of Hindman's theorem.

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[6] This notation comes from a topological perspective on the set of ultrafilters.

[7] One has to show that this actually defines an ultrafilter.

[8] Note that the principal ultrafilter generated by 0 is idempotent. To prevent such an anomaly, here we exclude 0 from $\mathbb{N}$.

Suppose that one has an idempotent ultrafilter $\mathcal{U}$. If we let R and B denote the elements of $\mathbb{N}$ of color red and blue respectively, then exactly one set, say R, belongs to $\mathcal{U}$. By the previous proposition, there is an infinite set $X \subseteq \mathbb{N}$ such that $\mathrm{FS}(X) \subseteq R$. This proves Hindman's theorem, *assuming* the existence of an idempotent ultrafilter. Do idempotent ultrafilters exist? Thankfully, again the answer is yes, and in abundance. The result relies on *Ellis' Lemma*, which says that, under certain conditions, which are fulfilled by the semigroup $\beta\mathbb{N}$, idempotent elements always exist. In fact, using Ellis' Lemma, one can even prove that, given any set A which contains $\mathrm{FS}(X)$ for an infinite set X, there is an idempotent ultrafilter $\mathcal{U}$ with $A \in \mathcal{U}$. Thus, if $A = B \cup C$ for some sets B and C, then one of them also contains $\mathrm{FS}(Y)$ for some infinite set Y, giving a stronger version of Hindman's theorem.

4 Erdős' $B + C$ conjecture

The above applications of ultrafilters were in the area of Ramsey theory. There have also been a number of applications to a different part of combinatorics known as *combinatorial number theory.* In this section, we discuss a striking recent result in this direction which resolved an old conjecture of Paul Erdős (1913-1996), who was one of the greatest mathematicians of the 20th century.

Theorem 9 (Donaldson–Moreira–Richter (2018)) *Suppose that $A \subseteq \mathbb{N}$ is of "positive density". Then there are infinite sets $B, C \subseteq \mathbb{N}$ such that A contains $B + C := \{b + c : \ b \in B, \ c \in C\}$.*

There are many notions of positive density for sets of numbers, all of which try to capture the idea that a set is of positive density if it contains a "positive proportion" of the numbers. For example, under any reasonable notion of density, the set of even numbers should have density $\frac{1}{2}$. Erdős' original conjecture says that for an infinite set $A \subseteq \mathbb{N}$ with positive *lower density* there are infinite sets $B, C \subseteq \mathbb{N}$ such that A contains $B + C$. Note that of all of the positive density assumptions, the lower density is one of the strongest. In the above theorem, the authors merely assume that A has positive *Banach density* (to be defined below), which is perhaps one of the weakest positive density assumptions, making the theorem even more impressive. The first major progress on Erdős' conjecture, due to Di Nasso, Jin, Leth, Lupini, Mahlburg, and the author, was to prove (using ultrafilter techniques) that the conjecture was true under the assumption that A has Banach density larger than $\frac{1}{2}$, that is, in some sense, A contains more than half of the numbers.

Before giving an idea behind the proof of Theorem 9, let us briefly describe a reason why ultrafilters might be involved. Recall that we defined an addition operation $\oplus$ on $\beta\mathbb{N}$. It turns out that $\oplus$ is *very* noncommutative, that is, most of the time, $\mathcal{U} \oplus \mathcal{V} \neq \mathcal{V} \oplus \mathcal{U}$. The next proposition shows the connection between Erdős' conjecture and ultrafilters.

Proposition 10 (Di Nasso) *Given $A \subseteq \mathbb{N}$, the following are equivalent:*

- *There are infinite subsets $B, C \subseteq \mathbb{N}$ such that A contains $B + C$.*
- *There are nonprincipal ultrafilters $\mathcal{U}$ and $\mathcal{V}$ such that $A \in (\mathcal{U} \oplus \mathcal{V}) \cap (\mathcal{V} \oplus \mathcal{U})$.*

We now briefly explain how ultrafilters are actually used in the proof of Theorem 9. First, we note the following.

Lemma 11 *Let $A \subseteq \mathbb{N}$. Suppose that there are $e_1 < e_2 < e_3 < \ldots$ and $L := \{l_1 < l_2 < l_3 < \ldots\}$ such that for all n:*

1. *$l_1 + e_n \in A$, $l_2 + e_n \in A$, $\ldots$, $l_n + e_n \in A$, and*
2. *There are infinitely many $l \in L$ with the property that $e_1 + l \in A$, $e_2 + l \in A$, $\ldots$, $e_n + l \in A$.*

Then there are infinite sets B and C such that A contains $B + C$.

The proof of Lemma 11 is a nice exercise that we leave to the reader. The next step in the proof is to rephrase Lemma 11 in terms of the density of the set A. In order to do so, we introduce in more details the notion of density. Consider a sequence $\mathcal{I} := (I_n)$ of intervals in $\mathbb{N}$. By an interval in $\mathbb{N}$, we mean a set of the form $[a, b] := \{n \in \mathbb{N} : \ a \leq n \leq b\}$, for $a, b \in \mathbb{N}$, with length given by the number of elements it contains, that is $|[a, b]| := b - a + 1$. We assume that the length of I_n goes to ∞ as n goes to ∞. For such a sequence of intervals $\mathcal{I}$, we define

$$d_{\mathcal{I}}(A) := \lim_{n \to \infty} \frac{|A \cap I_n|}{|I_n|},$$

whenever the limit exists. The idea is that we are trying to measure how big A is by taking "samples" of A from the interval I_n and asking what proportion of I_n lies in A. The limit means that we are considering this proportion for larger and larger intervals I_n. For example, declaring that $d_{\mathcal{I}}(A) = \frac{1}{100}$ roughly means that, for large enough n, approximately $\frac{1}{100}$ of the elements of I_n live in A. Notice that $d_{\mathcal{I}}(A) > 0$ means that A is "non-negligible" with respect to the sequence $\mathcal{I}$ but could be incredibly "sparse" for other intervals in $\mathbb{N}$. The *Banach density* of A is the largest value one gets by considering the numbers $d_{\mathcal{I}}(A)$ for various sequences $\mathcal{I}$. We now rephrase Lemma 11 with the vocabulary we just introduced.

Lemma 12 *Let $A \subseteq \mathbb{N}$. Suppose that there is $\epsilon > 0$, a sequence $\mathcal{I}$ as above, $e_1 < e_2 < e_3 < \ldots$ and $L := \{l_1 < l_2 < l_3 < \ldots\}$ such that for all n:*

1. *$l_1 + e_n \in A$, $l_2 + e_n \in A$, …, $l_n + e_n \in A$, and*
2. *$d_{\mathcal{I}}((A - e_n) \cap L) > \epsilon$.*

Then there are infinite sets B and C such that A contains $B + C$.

So the second property in Lemma 11 is now replaced by simply asking that, for a non-negligible number of l in L, $l + e_n \in A$. The proof of Lemma 12 now uses a result of Bergelson in order to pass to a subsequence of the numbers (e_n) and (l_n).

We can now get ultrafilters involved with yet another rephrasing of the previous lemma.

Lemma 13 *Let $A \subseteq \mathbb{N}$. Suppose that there is $\epsilon > 0$, a sequence $\mathcal{I}$ as above, and a nonprincipal ultrafilter $\mathcal{U}$ such that, for $\mathcal{U}$-almost all n, we have*

$$d_{\mathcal{I}}((A - n) \cap (A - \mathcal{U})) > \epsilon.$$

Then there are infinite sets B and C such that A contains $B + C$.

To see that the hypothesis of Lemma 13 implies the hypothesis of Lemma 12, one lets $L = A - \mathcal{U}$ and constructs a sequence $e_1 < e_2 < \ldots$ as follows. Let $l_1 < l_2 < \ldots$ enumerate the elements of L in increasing order. Since $A - l_1 \in \mathcal{U}$, there is $e_1 \in A - l_1$ such that $d_{\mathcal{I}}((A - e_1) \cap L) > \epsilon$. Since $A - l_2 \in \mathcal{U}$ and $\mathcal{U}$ is nonprincipal, there is $e_2 > e_1$ such that $e_2 \in (A - l_1) \cap (A - l_2)$ and for which $d_{\mathcal{I}}((A - e_2) \cap L) > \epsilon$. One constructs in this manner the sequence $e_1 < e_2 < \ldots$ and then proves that the hypothesis of the previous Lemma 12 holds if there is $\mathcal{I}$ for which $d_{\mathcal{I}}(A) > 0$. This proof is very complicated and we do not provide further details.

5 Further reading

The results mentioned in this snapshot are only a small sample of the main results in Ramsey theory and combinatorial number theory that are proven with ultrafilter methods. The books [2] and [4] are good references for further reading. A nice survey article on this topic, containing many other examples, is [1]. The algebraic properties of $\oplus$ on $\beta\mathbb{N}$ are quite fascinating (not to mention quite bizarre) and can be found in [3], which also contains a plethora of further combinatorial consequences.

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