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Truncated fusion rules for supergroups

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In the ’70s, physicists introduced a new type of symmetry – supersymmetry – to address some unresolved issues in particle physics models. Its mathematical foundations involve the representation theory of the associated symmetry groups, called supergroups.

Our aim is to understand fusion rules, which describe how a combination of two physical systems can be broken down into more fundamental building blocks. Although the answer is largely unknown, we can get approximate answers in some cases.

1 Symmetries and groups

The concept of symmetry is one of the foundational principles in mathematics and physics. A *symmetry* of a system is a transformation that leaves the system unchanged, or *invariant*. For example, a sphere in three-dimensional space looks the same after a rotation by an arbitrary angle, hence it is symmetric with respect to rotations. Since an abstract set does not depend on the order of its elements, the set $\{1, 2, \ldots, n\}$ is symmetric with regards to arbitrary *permutations* (that is, re-orderings) of the numbers $1, 2, \ldots, n$.

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The symmetry transformations of a system form the associated *symmetry group*. In the case of the sphere, it is the *special orthogonal group* $\mathrm{SO}(3)$ of all rotations about the center; in the case of the set $\{1, \dots, n\}$, it is the permutation group S_n, also called *symmetric group*. Symmetries need not be geometric in nature, as the permutation example shows.

Other examples of symmetries arise in physics: Particles can have an inner symmetry called *spin*. Elementary particles can be divided into two families, according to their behavior under spin transformations: bosons (like the photon) and fermions (like the electron). In the '70s, physicists described a new conjectural form of symmetry: *Supersymmetry* is a symmetry transformation that can transform bosons into fermions and vice versa.

Laying the mathematical foundations of supersymmetry has been an ongoing process since then. A key part of this is the representation theory of *supergroups*. In this area, we try to understand a mathematical problem that would physically correspond to the fusion of two physical systems.

Let us step back for a moment and return to the basic notions of a group. If we look at the set of rotations, it has a few remarkable properties:

1. If we take two rotations φ_1, φ_2, we get another rotation $\varphi_1 \circ \varphi_2$ by doing the rotations consecutively.
2. We can rotate by zero degrees; this rotation leaves all points unchanged.
3. For any rotation with angle θ there is an inverse rotation, namely by $-\theta$. If we first do one and then the other, we rotate by zero degrees in total.

In abstract terms, we have a set of transformations (the rotations) in which any two elements can be composed to yield another element from this set; there is a neutral element with respect to this composition (the rotation by zero degrees) and an inverse (the rotation by the angle $-\theta$). A set with such a composition is called a *group*. Groups abound in mathematics; an obvious example is the set of real numbers $\mathbb{R}$ with the usual addition: $a + b$ is another real number, the neutral element is 0, and the inverse to a is $-a$. Yet another example is the symmetric group or permutation group mentioned above.

2 Representations of groups

Representation theory studies groups – or other similar structures – by representing them as linear transformations on a different structure, called vector space. The prototypical example of a vector space is the three-dimensional space $\mathbb{R}^3$ or its generalization, the n-dimensional space $\mathbb{R}^n$. An element in $\mathbb{R}^3$ is given by a triple (x, y, z) or (x_1, x_2, x_3), where x_1, x_2, and x_3 are real numbers called *coordinates*. It is common to call these triples vectors.

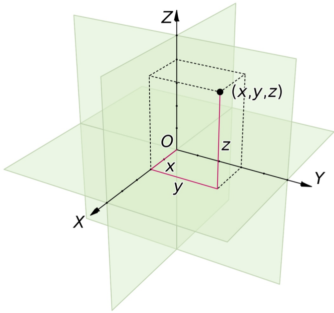

Figure 1: A point in $\mathbb{R}^3$ is defined by three coordinates x, y, z.

But $\mathbb{R}^3$ has more structure than just a set of vectors: We can add two triples,

$$(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3),$$

we can multiply them by real numbers,

$$a \cdot (x_1, x_2, x_3) = (a\,x_1, a\,x_2, a\,x_3) \quad \text{for } a \in \mathbb{R},$$

and there is the *zero vector* $(0, 0, 0)$, which fulfills

$$(0, 0, 0) + (x_1, x_2, x_3) = (x_1, x_2, x_3).$$

A structure with these properties is called a *vector space* in mathematics. The vector space $\mathbb{R}^3$ is specified by three coordinates, but there are vector spaces that cannot be described by a finite number of coordinates, therefore called infinite-dimensional.

A *linear map* $\varphi : V \to V$ of a vector space V is one that is compatible with the addition and scalar operation:

$$\varphi(x + y) = \varphi(x) + \varphi(y) \quad \text{and} \quad \varphi(a\,x) = a\,\varphi(x) \quad \text{for all} \quad x, y \in V, a \in \mathbb{R}.$$

Geometrically, this means that straight lines get mapped to straight lines, but they can be rotated and angles may be distorted. A *representation* of a group G then assigns to every element $g \in G$ a linear map $\varphi_g \colon V \to V$. We often say that G *acts on* V (via linear maps) and write $g \cdot v$ for the image of v under φ_g.

As linear maps are easier to understand than arbitrary groups, a representation of G allows us to study G by looking at the associated linear maps. By doing so, we lose information about the group. We should therefore understand the whole collection of representations.

3 A quick primer on representation theory

The study of representations of groups (or similar algebraic structures) is called representation theory. It is a vast theory with many different flavors, but at the heart of the matter there are some simple questions such as: How many representations are there? Can we classify them? In this generality, there is no hope to answer them. A first very harsh restriction might be to look only at finite-dimensional vector spaces. As a second step, one should not hope to get uniform answers for all groups, but restrict to particular groups of interest (such as S_n or $\mathrm{SO}(n)$). Thirdly, the problem of describing all finite-dimensional representations should be reduced to some basic building blocks.

Like all matter is composed of atoms, and all atoms of elementary particles, each representation should be built from certain fundamental building blocks. These building blocks are called *irreducible representations.* To fully understand representations, it is enough to describe the irreducible ones and the ways in which they can be combined to form other representations. A representation is called irreducible if it does not contain any non-trivial subrepresentation. This means there is no smaller vector space U inside of V such that the representation maps all vectors in U to vectors in U.

Let us look at the following action of the permutation group S_3 on $\mathbb{R}^3$: An element in $\mathbb{R}^3$ is given by a triple (x, y, z) of real numbers. If $\sigma \in \mathrm{S}_3$ is the permutation swapping the second and third entries, then φ_σ maps (x, y, z) to (x, z, y). In this way, we can specify a linear map from $\mathbb{R}^3$ to $\mathbb{R}^3$ for each element in S_3, which gives a representation of S_3 on $\mathbb{R}^3$.

But this representation is not irreducible! Indeed, look at the vectors in $\mathbb{R}^3$ where every entry is the same number, for example $(1, 1, 1)$. The space of such tuples is a one-dimensional subspace of $\mathbb{R}^3$, since we can identify it with $\mathbb{R}$ via the map $x \mapsto (x, x, x)$. This subspace is invariant under S_3, as swapping identical entries does not change (x, x, x). We have hence found a subrepresentation in $\mathbb{R}^3$, so our original representation is not irreducible.

The set of tuples (x, y, z) that satisfy $x + y + z = 0$ is also a vector space and invariant under S_3, as reordering the entries does not change their sum. This space, together with the permutation action, is sometimes called the *standard representation st* of S_3. As a subset of $\mathbb{R}^3$, it is the two-dimensional plane orthogonal to the vector $(1, 1, 1)$. Swapping two entries gives a reflection about some line in this plane, while the permutation $(x, y, z) \mapsto (y, z, x)$ is a rotation about $120°$.

Any vector in $\mathbb{R}^3$ is a sum of vectors from the two subrepresentations (because $(x, y, z) = (m, m, m) + (x - m, y - m, z - m)$ for $m = \frac{x+y+z}{3}$) and they have only the zero-vector in common. We write this as

$$\mathbb{R}^3 \cong \mathbb{R} \oplus st \quad \text{(as representations of } \mathrm{S}_3\text{)}.$$

The same is true if we start with an arbitrary number n of dimensions instead of 3. In this case, the standard representation has the dimension $n-1$. It can be checked that st is irreducible, hence the representation of S_n on $\mathbb{R}^n$ is the direct sum of two irreducible representations.

In fact, Maschke's Theorem asserts that every finite-dimensional representation of a finite group on a real or complex vector space can be written as a direct sum of irreducible representations. This theorem is no longer true if one works with infinite groups like $\mathbb{R}$ or infinite-dimensional vector spaces.

4 Sums, products, and fusion rules of representations

The representation theory of the permutation groups is an old classical subject. Much of it goes back to the work of Isaai Schur and Georg Frobenius more than a hundred years ago.

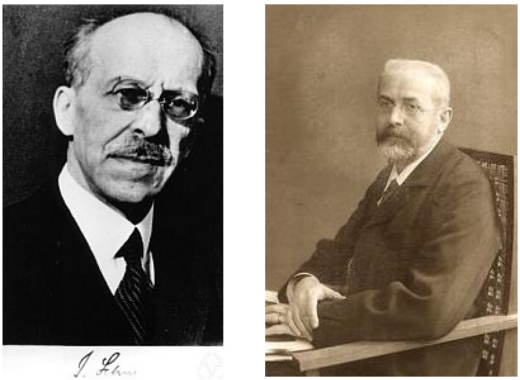

Figure 2: Isaai Schur (1875–1941) and Georg Frobenius (1849–1917).

But even here, one encounters elementary, yet unsolved questions. One such question concerns the *fusion rules.* Given two representations V, W of a group G (this means that V and W are vector spaces on which G acts via linear maps), we can use these to construct new representations. One such construction is the *direct sum* $\oplus$ used above.

Writing $V \oplus W$ means looking at the tuples (v, w) where $v \in V,\ w \in W$. These tuples are closed under addition $(v_1, w_1) + (v_2, w_2) = (v_1 + v_2, w_1 + w_2)$ and multiplication by a real number $a \cdot (v, w) = (a\,v, a\,w)$, and hence form a vector space of dimension $\dim(V) + \dim(W)$. The vector space $V \oplus W$ is a representation of G if we define $g \cdot (v, w) = (g \cdot v, g \cdot w)$. One example of such a construction is $\mathbb{R}^2$: As a vector space, this is simply $\mathbb{R} \oplus \mathbb{R}$.

Yet another way to build a new representation is the *tensor product* $V \otimes W$. This is again a new vector space, but this time of dimension $\dim(V) \cdot \dim(W)$. It contains elements $v \otimes w$ for $v \in V,\ w \in W$, but also more complicated expressions (sometimes called entangled states) of the form $v_1 \otimes w_1 + v_2 \otimes w_2 + \cdots + v_k \otimes w_k$ for some vectors v_i in V and some vectors w_j in W.

If we write the tensor product

$$V \otimes W = \underbrace{I_1 \oplus \ldots \oplus I_1}_{c_1 \text{ times}} \oplus \underbrace{I_2 \oplus \ldots \oplus I_2}_{c_2 \text{ times}} \oplus \ldots \oplus \underbrace{I_n \oplus \ldots \oplus I_n}_{c_n \text{ times}}$$

of representations V and W as a sum of representations $I_1, \ldots, I_n$, the numbers c_i, which count how often I_i turns up in this decomposition, are called *multiplicities*. A rule that tells us what these multiplicities are and how they are computed (and therefore how the tensor product $V \otimes W$ decomposes) is called a *fusion rule*.

In the situation of the permutation group, Maschke's Theorem states that any finite-dimensional representation can be written as a direct sum of irreducible representations. In this case the multiplicities c_i have a special name: they are called *Kronecker coefficients*. While the finite-dimensional representation theory of the permutation group over the real or complex numbers is classical and we know all irreducable representations, we do not know any good description of these coefficients yet! We would like to have a closed combinatorial expression of the c_i, but we do not know it. The lesson is that finding the fusion rules is going to be very hard in general if we already fail for such a well-studied example as S_n.

5 Continuous symmetries and Lie groups

It is important to take into account that there are many types of groups, and hence we cannot expect a single theory that describes all possible representations of groups. For example, groups could describe continuous symmetries (such as $\mathrm{SO}(3)$, where we can continuously vary rotation angles) or discrete symmetries (such as the permutation group S_n). Continuous symmetries lead to the theory of *Lie groups*, named after the Norwegian mathematician Sophus Lie (1842–1899). An analysis of their representations requires other methods than the study of discrete groups such as S_n.

The most important Lie groups are $\mathrm{GL}(n, \mathbb{R})$, the group of invertible linear maps from $\mathbb{R}^n \to \mathbb{R}^n$, and its analog $\mathrm{GL}(n, \mathbb{C})$, which is obtained by replacing the real numbers $\mathbb{R}$ with complex numbers $\mathbb{C}$. These groups are called *matrix groups*, because it is often convenient to represent their elements as invertible $n \times n$ *matrices*, which are arrays of real or complex numbers with n rows and n columns that have a special rule for multiplication. (For further reading on matrices, see Snapshot 5/2019 [2].) Many other important Lie groups occur naturally as subgroups:

- The *special linear groups* $\mathrm{SL}(n,\mathbb{R})$ and $\mathrm{SL}(n,\mathbb{C})$, consisting of linear maps whose determinant is one. This means that they may distort shapes but do not change volumes.
- The *special orthogonal* group $\mathrm{SO}(n)$, consisting of all rotations in $\mathbb{R}^n$, and the *orthogonal* group $\mathrm{O}(n)$, which additionally contains all combinations of reflections and rotations.
- The *unitary* groups $\mathrm{U}(n)$ and *special unitary* groups $\mathrm{SU}(n)$, which are analogous to $\mathrm{O}(n)$ and $\mathrm{SO}(n)$ for the complex vector space $\mathbb{C}^n$.
- Other notable examples, which we will not consider in detail, are the series of *symplectic* groups $\mathrm{Sp}(2n,\mathbb{C})$ and several "exceptional Lie groups".

The representation theory of Lie groups is an extremely rich subject that has connections to almost all areas of pure mathematics. In general, even a finite-dimensional representation over the real or complex numbers might not be a sum of irreducible representations. However, *Weyl's Complete Reducibility Theorem* states that such a decomposition is always possible for algebraic representations of matrix groups such as $\mathrm{GL}(n,\mathbb{R})$, $\mathrm{GL}(n,\mathbb{C})$, $\mathrm{SO}(n)$, $\mathrm{O}(n)$, $\mathrm{Sp}(2n,\mathbb{C})$, and $\mathrm{SU}(n)$. Contrary to the finite group case, there are always infinitely many irreducible representations, but they can often be classified. Coming back to our original problem, we can now ask how we can decompose the tensor product of two representations into a direct sum of simpler representations.

For certain groups of continuous symmetries, there are algorithmic descriptions of these fusion rules: the "Littlewood–Richardson rule" for $\mathrm{GL}(n)$, $\mathrm{SL}(n)$, and variants of it for other classical groups. In fact, Dudley Littlewood and Archibald Richardson formulated an algorithm for this problem in the $\mathrm{SL}(n)$-case in 1934, which was finally proven to be correct in the '70s by Marcel-Paul Schuetzenberger and Glânffrwd Thomas. In the words of Gordon James:

> *Unfortunately the Littlewood-Richardson rule is much harder to prove than was at first suspected. The author was once told that the Littlewood-Richardson rule helped to get men on the moon but was not proved until after they got there.*

Indeed, as soon as the dimensions of the representations become large, their tensor product decomposes into zillions of summands with no obvious pattern.

6 Algebraic aspects of the Standard Model

Physicists are often not interested in all possible groups, they mostly need very special Lie groups: $\mathbb{R}$ and $\mathrm{U}(1)$ for *translation symmetry*, $\mathrm{SO}(3)$ for *spatial* *rotations*, SU(2) to describe *isospin*, and a few others more. The most important groups are those that arise as "gauge groups" in gauge field theories, notably the "Standard Model", which is a special kind of quantum field theory and currently the most successful framework in particle physics.

The symmetry group of the Standard Model and the classification of elementary particles are based on a feedback loop between symmetry considerations (that is, the representation theory of possible symmetry groups) and empirical data. Analysis of data from high energy collision experiments suggested conservation laws and symmetry constraints, which in turn have led to the prediction of new particles that could ultimately be found in experiments. One of the most recent examples is the Higgs boson, which was predicted by Peter Higgs and others in 1964 and experimentally confirmed in 2012.

Ultimately, the goal is to describe all matter in terms of elementary particles and the interactions between them by fundamental forces. The Standard Model achieves this for all known elementary particles and forces except for gravity.

The wave-like nature of elementary particles is modelled by vector fields, which assign a vector to every point in space and time. The dynamics of the Standard Model (that is, how the vector fields evolve and oscillate over time) are described by a *Lagrangian* $\mathcal{L}$, a mathematical expression from which all further equations of the theory can be derived. For example, if the Lagrangian contains a product of fields, the corresponding particles interact with each other.

To get this Lagrangian, one first postulates a set of symmetries and then tries to find the most general Lagrangian that satisfies these symmetries. As in all field theories that respect Albert Einstein's theory of relativity, the laws of

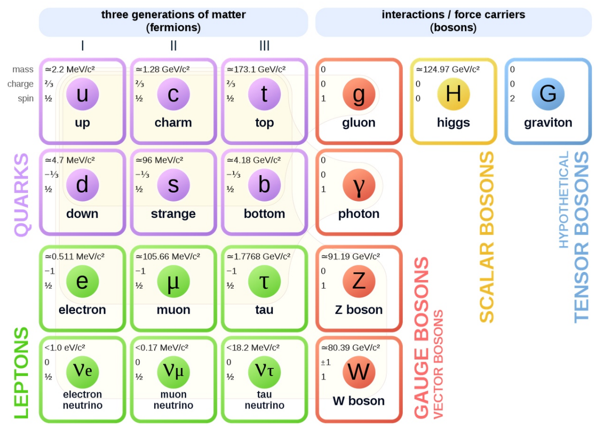

Figure 3: The elementary particles in the Standard Model.

physics must remain unchanged under changes in speed, position, or rotations of the coordinate system. These transformations form the *Poincaré group*, so the Lagrangian has to be symmetric under this group.

Additionally, the Lagrangian has an internal symmetry ("local gauge symmetry") with respect to the group $G = \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$. These three groups correspond to the three interactions that the Standard Model incorporates: electromagnetism, weak, and strong nuclear force.

Each of the elementary particles (like the electron or any of the quarks) is represented by a vector field on which one of these three groups acts as a symmetry group. An example is given by the up-quark: It comes in three different polarizations (sometimes called red, green, blue) and each polarization is described by one complex number. Together, they combine to the standard representation $\mathbb{C}^3$ of $\mathrm{SU}(3)$ (when ignoring the $\mathrm{SU}(2) \times \mathrm{U}(1)$-part).

As mentioned above, the Lagrangian is an expression that combines the values of these vector fields and is also symmetric under these groups. The evolution of a physical state in the Standard Model is given by a so-called *path integral* (or *Feynman integral*) over a term involving the Lagrangian, and is far beyond this little paper. These Feynman integrals are not mathematically rigorously defined, but they have been used by physicists since decades to calculate the effect of collisions and other physical effects with very high precision.

If we want to describe the collision of two particles, the result depends on the fusion rules between the corresponding irreducible representations. While

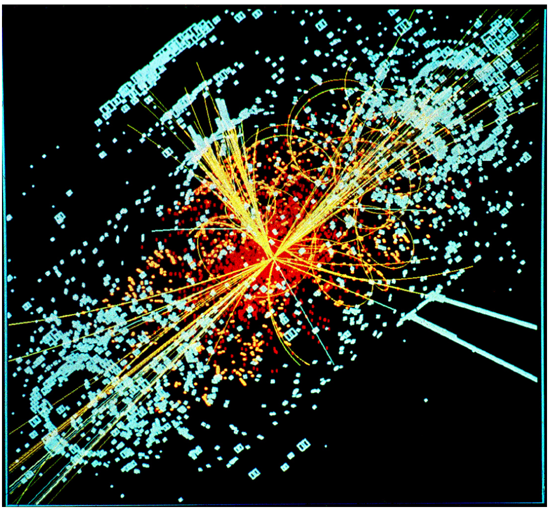

Figure 4: Data from a particle collision.

there are many open questions about Feynman integrals, the part that involves the fusion rules is a classical piece of mathematics and well-understood. The situation changes considerably if one tries to replace the Standard Model and its gauge group by a more complicated (but perhaps more elegant) theory based on the notion of *supersymmetry*.

7 Super structures

The Standard Model of particle physics had tremendous success in unifying electromagnetism, weak, and strong nuclear force, and agrees with experimental results. However, it has several shortcomings, which lead physicists to search for alternatives. In particular, supersymmetric extensions of the Standard Model provide elegant solutions to some of these problems. However, no experimental evidence for such an extension was found at the LHC or other colliders so far, and hence the concept of supersymmetry remains in limbo despite its theoretical advantages.

Mathematically, the passage to the supersymmetric extension involves replacing the symmetry groups of the model, like the gauge group $\mathrm{U}(1)\times\mathrm{SU}(2)\times\mathrm{SU}(3)$ or the Poincaré group, with a Lie supergroup. This is a group that has an *even part* (corresponding to boson particles), an *odd part* (corresponding to fermion particles), and obeys some additional rules. Similarly, the vector spaces on which these groups or algebras act are replaced by *super vector spaces*, vector spaces with an even and an odd part. The easiest example is simply

$$\mathbb{C}^{m|n} = \mathbb{C}^m \oplus \mathbb{C}^n,$$

a vector space with two parts, where $\mathbb{C}^m$ is seen as even and $\mathbb{C}^n$ as odd.

The space of linear maps of $\mathbb{C}^{m|n}$ to itself also has an even and odd part: A transformation $\mathbb{C}^{m|n} \to \mathbb{C}^{m|n}$ is called even if it maps the even part to the even part and the odd part to the odd part. It is called odd if it maps the even to odd part and vice versa.[2] Hence the space of linear maps of $\mathbb{C}^{m|n}$ is itself a super vector space! Mimicking the definition of $\mathrm{GL}(n,\mathbb{C})$, we define

$$\mathrm{GL}(m|n,\mathbb{C}), \quad \textit{the General Linear Supergroup}$$

to be the group of invertible linear maps $\mathbb{C}^{m|n} \to \mathbb{C}^{m|n}$. The supergroup incorporates the classical one via $\mathrm{GL}(m|0,\mathbb{C}) = \mathrm{GL}(m,\mathbb{C})$. There are analogs of the classical groups $\mathrm{SO}(n)$ and $\mathrm{Sp}(2n)$ (the "orthosymplectic supergroups"), but also new types of groups that have no classical counterpart.

[2] Note that not every linear transformation is even or odd. But every transformation can be written as a sum of an even and an odd transformation, just as every vector in $\mathbb{C}^{m|n}$ is the sum of an even and an odd vector.

Similar to the classical theory of Lie groups, one can now ask: What are the irreducible representations? What are their dimensions? Can every representation be written as a direct sum of irreducible representations? These and further questions have been investigated since the foundational work of Victor Kac [7] in the '70s; and the study of (algebraic) representations of Lie supergroups has now become a thriving area in pure mathematics with connections to many other fields such as algebraic geometry, quantum topology, and even analytic number theory.

8 Fusion rules for supergroups and truncations

One major difference between classical Lie groups and Lie supergroups is that Weyl's Theorem fails for supergroups: Not all finite-dimensional representations of supergroups can be written as a sum of irreducible representations. This means that we will encounter representations that have many subrepresentations, but there is no way to split them into a direct sum of two other representations. Such representations are called *indecomposable*[3]. The occurence of such representations renders many tools from classical Lie theory useless and the answers to the questions posed above are much more complicated or are even unknown.

In the beginning, there is a fortunate surprise: It is quite easy to parametrize all irreducible representations for many concrete Lie supergroups. In the case of $\mathrm{GL}(m|n)$, we can parametrize all irreducible representations by $m+n$ integer numbers. This means that there is an irreducible representation called

$$L(\,\lambda_1,\dots,\lambda_m \mid \lambda_{m+1},\dots,\lambda_{m+n}\,)$$

for any two lists of m integers $\lambda_1,\dots,\lambda_m$ and n integers $\lambda_{m+1},\dots,\lambda_{m+n}$, sorted in descending order. We abbreviate this irreducible representation by $L(\lambda)$ and call the two lists of λs the *weights* of this representation. Every irreducible representation of $\mathrm{GL}(m|n)$ is then of the form $L(\lambda)$ or $\Pi L(\lambda)$, where Π means that we swap the even and the odd part of the underlying super vector space.

Many of the questions that one can ask about these $L(\lambda)$s turn out to be very hard: What is the dimension of $L(\lambda)$? What are the dimensions of its even and odd parts? In what ways can we combine such $L(\lambda)$ to form these big complicated indecomposable representations? And, most importantly for this article, what are the fusion rules for $L(\lambda)\otimes L(\mu)$? Some of these questions have been answered in the last 10–20 years [8, 1, 4] but there are still open questions, and the situation is even more complicated for other Lie supergroups.

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[3] Remember that a representation is called irreducible if it contains no smaller subrepresentations, while indecomposable means that it is not a sum of some of its subrepresentations. Every irreducible representation is therefore indecomposable, but not vice versa.

Quite generally, the fusion rules are known for the case $\mathrm{GL}(m|1)$, but beyond that, the decomposition of $L(\lambda) \otimes L(\mu)$ is only known for special λ and μ.

It was suggested in [3] and carried out in [5, 6] that one should instead look at *truncated fusion rules*: Our goal of finding a decomposition

$$L(\lambda) \otimes L(\mu) = I_1 \oplus I_2 \oplus \ldots \oplus I_n$$

of a tensor product into a sum of indecomposable summands $I_1, \ldots I_n$ becomes more tractable if we disregard all summands whose even and odd parts have the same dimension. The difference of the dimensions of the even and odd part of a super vector space is called its *superdimension*, so we ignore all summands with superdimension zero when we truncate the decomposition.

While the actual fusion rules are unknown, it turns out that one can determine these truncated fusion rules in almost all cases!

Our main result [5] essentially says that the fusion rule describing this decomposition is the same as for classical groups such as $\mathrm{SL}(n)$, $\mathrm{SU}(n)$, $\mathrm{Sp}(2n)$, and so on!

The key point is that for each irreducible representation $L(\lambda)$ of $\mathrm{GL}(m|n)$, we can attach a group H_λ (which is now not a supergroup, but a classical group like those above) and an irreducible representation $V(\lambda)$ of H_λ such that $L(\lambda) \otimes L(\mu)$ decomposes exactly as $V(\lambda) \otimes V(\mu)$ decomposes. We also described how to calculate the groups H_λ and the representations $V(\lambda)$ explicitely.

How does this work in practice? Suppose we have $\lambda = \mu = (2, 1, 0 \mid 0, \text{-}1, \text{-}2)$ and look at the representation $L(\lambda)$ of $\mathrm{GL}(3|3)$. Our main theorems tell us that the corresponding pair is $H_\lambda = \mathrm{Sp}(6)$ and $V(\lambda) = L(1, 0, 0)$. Here, $L(1, 0, 0)$ is actually just the standard representation of $\mathrm{Sp}(6)$: The vector space $\mathbb{C}^6$ with the action of $\mathrm{Sp}(6)$ by linear maps. The classical fusion rules for $\mathrm{Sp}(6)$ tell us that $L(1, 0, 0) \otimes L(1, 0, 0)$ decomposes as

$$L(1, 0, 0) \otimes L(1, 0, 0) \cong L(0, 0, 0) \oplus L(2, 0, 0) \oplus L(1, 1, 0).$$

To each of the three summands corresponds an indecomposable representation I_1, I_2, I_3 of $\mathrm{GL}(3|3)$ whose superdimension (that is, the difference of the dimensions of the even and odd part) agrees with the dimension of the $\mathrm{Sp}(6)$-representations. Hence

$$L(2, 1, 0 \,|\, 0, \text{-}1, \text{-}2) \otimes L(2, 1, 0 \,|\, 0, \text{-}1, \text{-}2) \cong I_1 \oplus I_2 \oplus I_3,$$

up to summands of superdimension zero. Now we can iterate this further and apply this to tensor products between the indecomposable summands that appear in this way. In the above example, I_3 corresponds to $L(1, 1, 0)$. In order to compute $I_3 \otimes I_3$ up to superdimension zero, we can look at $L(1, 1, 0) \otimes L(1, 1, 0)$ and match the resulting summands with indecomposable summands in $I_3 \otimes I_3$ and so forth.

Alas, there is a small caveat: For some very special λ, we cannot determine H_λ completely. In these cases, we have two candidates for H_λ and cannot decide which of the two is actually the right one.

9 Back to physics?

Let us recap what we have discussed so far: For some important types of groups, there are theorems that ensure representations can be decomposed into sums of simpler, irreducible representations. However, calculating these decompositions in practice is much more difficult. Fusion rules, which describe how tensor products are decomposed, are only known for a few special groups. In physics, these fusion rules are particularly important for understanding the interactions between multiple particles or physical systems.

Supergroups have an even more complicated representation theory. But for certain supergroups G, we can now associate a classical group H_λ to every representation $L(\lambda)$ of G and calculate truncated fusion rules of G from the fusion rules of H_λ, where "truncated" means that we ignore the summands of superdimension zero. While these results can be seen as a first step to obtain general fusion rules, one may wonder whether there is more to it, whether maybe the truncation actually has a physical meaning?

Let us suppose we have a supersymmetric extension of the Standard Model of particle physics, in which the symmetry group is replaced by a supergroup G. If the vector fields that correspond to supersymmetric particles take values in a finite-dimensional representation V of G, the computation of interactions between particles require the analysis of higher tensor products of V. These are approximated by our truncated fusion rules.

A particular important case is the one of $G = \mathrm{GL}(4|4)$, due to its connection to the *super conformal group*, a generalization of the conformal group, which consists of those transformations of four-dimensional spacetime (three space dimensions and one time dimension) that preserve angles.

Here is a list of the groups associated to some small representations of $\mathrm{GL}(4|4)$, where Sp^c stands for the compact symplectic group:

Representation $L(\lambda)$	superdimension	associated group H_λ
$L(3,2,1,0 \mid 0,-1,-2,-3)$	$\mathrm{sdim} = 24$	$H_\lambda = \mathrm{SO}(24)$ (conjecturally)
$L(3,2,0,0 \mid 0,0,-2,-3)$	$\mathrm{sdim} = 12$	$H_\lambda = \mathrm{SU}(12)$
$L(3,1,1,0 \mid 0,-1,-1,-3)$	$\mathrm{sdim} = 12$	$H_\lambda = \mathrm{Sp}^c(12)$
$L(3,1,0,0 \mid 0,0,-1,-3)$	$\mathrm{sdim} = 8$	$H_\lambda = \mathrm{SU}(8)$
$L(3,0,0,0 \mid 0,0,0,-3)$	$\mathrm{sdim} = 4$	$H_\lambda = \mathrm{SU}(4)$
$L(2,2,1,0 \mid 0,-1,-2,-2)$	$\mathrm{sdim} = 12$	$H_\lambda = \mathrm{SU}(12)$

$L(\,2,2,0,0 \mid \ \ 0,\ \ 0,\text{-}2,\text{-}2\,)$	sdim =	6	$H_\lambda = \mathrm{SO}(6)$
$L(\,2,1,1,0 \mid \ \ 0,\text{-}1,\text{-}1,\text{-}2\,)$	sdim =	8	$H_\lambda = \mathrm{SU}(8)$
$L(\,2,1,0,0 \mid \ \ 0,\ \ 0,\text{-}1,\text{-}2\,)$	sdim =	6	$H_\lambda = \mathrm{Sp}^c(6)$
$L(\,2,0,0,0 \mid \ \ 0,\ \ 0,\ \ 0,\text{-}2\,)$	sdim =	3	$H_\lambda = \mathrm{SU}(3)$
$L(\,1,1,1,0 \mid \ \ 0,\text{-}1,\text{-}1,\text{-}1\,)$	sdim =	4	$H_\lambda = \mathrm{SU}(4)$
$L(\,1,1,0,0 \mid \ \ 0,\ \ 0,\text{-}1,\text{-}1\,)$	sdim =	3	$H_\lambda = \mathrm{SU}(3)$
$L(\,1,0,0,0 \mid \ \ 0,\ \ 0,\ \ 0,\text{-}1\,)$	sdim =	2	$H_\lambda = \mathrm{SU}(2)$
$L(\,1,1,1,1 \mid \text{-}1,\text{-}1,\text{-}1,\text{-}1\,)$	sdim =	1	$H_\lambda = \mathrm{U}(1)$

In fact, in the first example we cannot rule a second possibility.

The reader will observe that the smallest arising groups are U(1), SU(2), and SU(3). One may ask whether the appearence of these three groups is a mere accident, or whether there does exist some connection with the symmetry group $\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$ of the Standard Model of elementary particle physics?

One special feature of supersymmetric field theories is that superpartners of particles sometimes have (almost) the same effect as the original particles but with a minus sign, so that the effects cancel each other. These cancellations are responsible for many of the more amenable aspects of such theories compared to the Standard Model. Due to such cancellations, it might happen that the contributions of summands of superdimension zero are too small to measure in the energy ranges reachable by particle physics experiments. Hence a physical observer might come up with the impression that the underlying rules of symmetry are imposed by the representation theory of the groups H_λ; and the groups H_λ or their product would appear as an internal symmetry group of the theory in an approximate sense.

Of course this is highly speculative. Whether there exists any supersymmetry in nature at all, and whether our H_λ appear as approximate symmetry groups of such a theory, will probably take many years to uncover.

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