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Vertex-to-Self Trajectories on the Platonic Solids

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We consider the problem of walking in a straight line on the surface of a Platonic solid. While the tetrahedron, octahedron, cube, and icosahedron all exhibit the same behavior, we find a remarkable difference with the dodecahedron.

1 The anti-social jogger: trajectories on a tetrahedron

Imagine a tetrahedron the size of a (small) planet. At each of the four corners of the tetrahedron is a house. In one of the houses lives a jogger. Each morning she sets off from her home jogging in a straight line. Her friendly neighbors interrupt her jog with a conversation if she runs by their houses. Can she run in a straight line, without ever changing direction, and return home, avoiding all of her neighbors? We will call this an *anti-social path*. Stated more mathematically, is it possible to draw a straight-line path on a tetrahedron that starts and ends at the same vertex and doesn't cross any other vertex?

Problems of this type have a long history, going back at least as far as 1906, to the work of Paul Stäckel (1862–1919) and Carl Rodenberg (1851–1933) of Hannover [6, 5]. They were motivated by "singularities" in equations that define *geodesics* on surfaces, where a geodesic means the shortest path between two points. An issue that they considered was how to define straight line trajectories

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as we cross various boundaries. Inside a face of the tetrahedron, it is clear how to travel in a straight line. To travel over an edge, Stäckel observed that there is a unique way to continue walking in a straight line over an edge, as long as you are not at a vertex. A way to see this is to consider the *net* of the tetrahedron: a flat diagram with folding instructions that, when assembled, forms the tetrahedron, as in Figure 1.

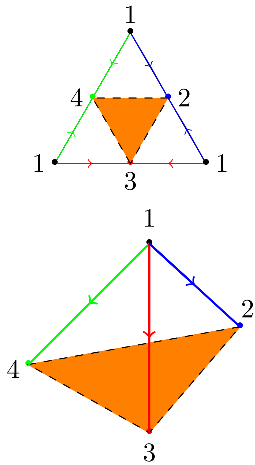

Figure 1: A net for the tetrahedron. Below we see it folded up with the orange face on the bottom.

Drawing a straight line on the net, and then assembling the net, the straight line remains and shows how to cross edges. That is, a line is determined by the initial direction – unless it hits a vertex. What happens then was debated between Stäckel and Rodenberg, and we (along with most modern mathematicians) choose the convention that there is no unique way to continue walking when you reach the corner of a face. Since any point in the interior of a flat region has 360 degrees around it, a line passing through it cuts this $360°$ angle into two pieces of 180 degrees on each side of it. The vertex of a tetrahedron has three triangles coming together, for a total angle of 180 degrees, which is too small to split into $180° + 180°$, making the notion of going straight ambiguous. So if we reach the vertex of a tetrahedron, we stop.

The question remains: can the anti-social jogger come back home without being stopped by a neighbor? Using our net, and an "unfolding" trick, we'll show that on the tetrahedron, the answer is no. That is, our jogger will either have to turn, or to visit a neighbor, to come back home.

Let's return to our net, and notice that when we assemble the tetrahedron, we are gluing sides of the same color by rotation of 180°. The unfolding trick, introduced by Fox-Kershner and Zelmjakov-Katok, takes two copies of our original picture, rotates one by 180°, and glues the resulting figure by translation, as shown in Figure 2.

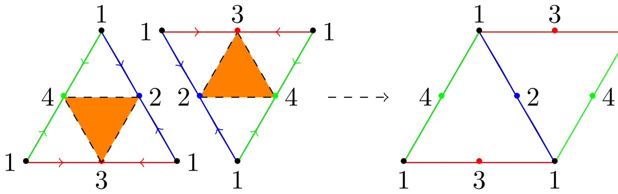

Figure 2: Unfolding the tetrahedron

We can further unfold this unfolding, to tile the plane. A portion of this tiling is shown below in Figure 3. This picture can be continued infinitely in all directions, tiling the full plane. The advantage of this perspective is that trajectories never change direction when drawn on the tiled picture. This is in contrast to drawing trajectories on the net for the tetrahedron where the direction could change by rotation by π, due gluing identified sides that are rotated by π, as one can see Figure 1. Our question now becomes: can we draw a straight line in this picture, starting and ending at a dot of the same color, without passing through a dot of any other color? We leave the verification that you cannot do this as an exercise for the interested reader.

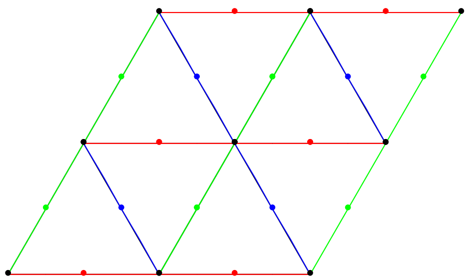

Figure 3: Unfolding the unfolding.

So the tetrahedron doesn't work, but could the jogger move to a different shaped planet?

2 Regular shapes: the Platonic solids

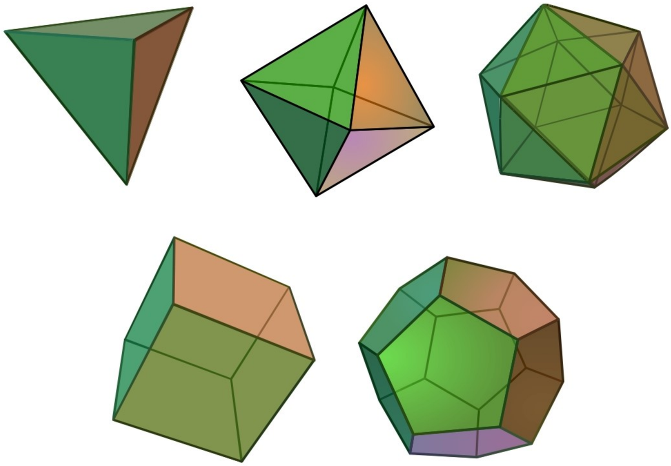

Figure 4: The five Platonic solids.

In kindergarten, we learn about the *regular polygons*: shapes whose sides are all the same: for example, the equilateral triangle, the square, the regular pentagon. There are infinitely many of these – for any number of sides, there is a *regular polygon* with that number of sides, where regular means that all the sides have the same length, and the angle at each corner is the same.

If we move up a dimension and ask for solids with identical regular polygons for faces, so all edges of the same length, and which have the same number of faces meeting at each corner, it turns out that there are only five such shapes. Together they are known as the *Platonic solids*. To get a closed solid, there must be at least three polygons at every corner of the solid. The sum of the angles of the polygons at each corner must be the same for all corners, and must be strictly less than 360°. With these restrictions, the only possible faces are triangles, squares, and pentagons because three hexagons meeting at a corner yields a total angle 360°, and the angle gets larger as the number of sides increases. Squares and pentagons can have at most three polygons meeting at a corner. For triangles, three, four, or five triangles are all possible. The five Platonic solids are illustrated in Figure 4.

3 Anti-social joggers: vertex-to-self trajectories

Returning to our jogger: is there a different Platonic solid-shaped planet she could move to?

Question: On which Platonic planets can a jogger avoid her neighbors?

This question was studied in [3], and continued in [4]. As we saw, the jogger on the tetrahedron has no anti-social path. This is also true on the octahedron, cube, and icosahedron: regardless of which direction she chooses, either she will never return home or she will run into one of her neighbors' houses. The argument for the other solids is similar to our argument for the tetrahedron; it is quite a bit more involved, but still relies on the fact that you can tile the plane with triangles or squares.

One might conjecture that since the answer is no for these four Platonic solids, it should also be no for the dodecahedron. However, the jogger on the dodecahedron is in luck! It is possible for her to run on an anti-social path; one possibility is illustrated in Figure 5.

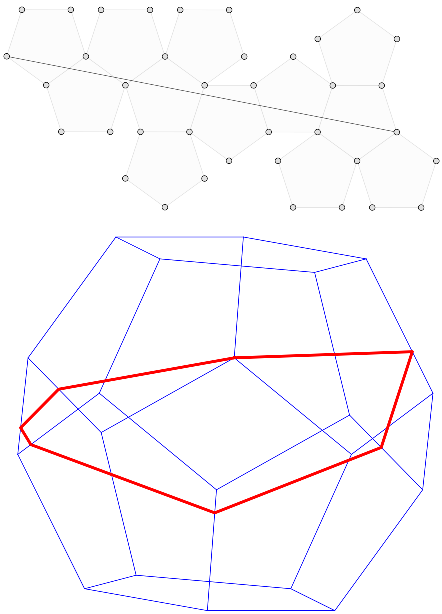

Figure 5: An anti-social jogging path on the dodecahedron, shown on a net and folded up.

As we hinted above, the crucial difference between the dodecahedron and the other solids is that the face of a dodecahedron, a regular pentagon, does not tile the plane.

4 Jogging paths

Happy that she can jog around her planet without seeing her neighbors, she grows tired of running the same path every day. Is there another path that she can take? If so, how many different anti-social paths are there?

In mathematics (and outside of it), we seek complete understanding of phenomena we observe. In this case, a complete understanding corresponds to having a list of all the possible anti-social trajectories. With Pat Hooper [2], we were able to solve this problem by using the theory of "lattice translation surfaces" to show that there are in fact infinitely many trajectories, where trajectories are allowed to self-intersect. In other words, while you walk around the dodecahedron planet, you can walk across a path on which you already walked.

Since there are infinitely many trajectories, what does it mean to make a list of infinitely many things? The key here is the notion of *equivalence classes.* For any set of mathematical objects, if there is a notion of equivalence defined on the set, we can split the elements up into subsets of those that are equivalent to each other. A good way to understand the idea of equivalence classes is the idea of even and odd numbers. There are infinitely many natural numbers: $1, 2, 3, \ldots$. Nevertheless, we can classify them as even, $2, 4, 6, \ldots$ and odd $1, 3, 5, \ldots$. Given a natural number, it is either even or odd. To represent these two classes, we can get every even (respectively odd) number by starting with 0 (respectively 1), and adding an even number to it.

Returning to the anti-social trajectories on the dodecahedron, we sort them into subsets using a natural type of symmetry. We found that there are 31 different types of trajectories from a vertex to itself when they are sorted by what are called "affine symmetries". An affine symmetry is essentially a way of deforming the dodecahedron by applying a linear transformation, and then cutting and reassembling it back into itself. Just like in the natural numbers example, any of the infinitely many trajectories from a vertex to itself can be placed into exactly one of these 31 different equivalence classes, and given a trajectory, it is possible to construct infinitely many distinct trajectories of the same type from that trajectory. That is, we can start with any trajectory, and by doing an affine symmetry construct every other trajectory of the same type. These affine symmetries are different than the usual symmetries of the dodecahedron – we do not distinguish between two trajectories which are images of each other by a usual symmetry. Finally, we should note that the group of affine symmetries is countably infinite, while there are only finitely many of the ordinary symmetries.

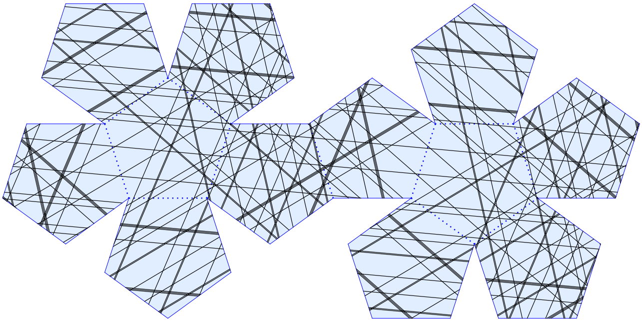

Figure 6: A really long vertex-to-self trajectory. This is a representative from equivalence class 25.

5 Further reading

We invite the reader to visit the website below in order to explore the interactive 3D animations with different trajectories. The website also contains nets of dodecahedra that can be used for grade school or math club activities. Two such activities are suggested in the document available for download on the website.

http://userhome.brooklyn.cuny.edu/aulicino/dodecahedron/

There is also the following video available where anti-social walks on the dodecahedron are explained and visualised:

https://www.numberphile.com/videos/yellow-brick-road

The reader is also encouraged to read the paper [1], which is accessible to a general audience. Technical details for specialists are contained in [2].

Image credits

Figure 4 https://commons.wikimedia.org/wiki/File:Platonic_solids.jpg

Figure 5 Taken from [1]

All other images created by the authors.

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